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ELEMENTS

OF

PLANE AND SOLID GEOMETRY.

BY

G. A. WENTWORTH, A. M.,

PROFESSOR OF MATHEMATICS IN PHILLIPS EXETER ACADEMY.

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PREFACE.

Most persons do not possess, and do not easily acquire, the power of abstraction requisite for apprehending the Geometrical conceptions, and for keeping in mind the successive steps of a continuous argument. Hence, with a very large proportion of beginners in Geometry, it depends mainly upon the *form* in which the subject is presented whether they pursue the study with indifference, not to say aversion, or with increasing interest and pleasure.

In compiling the present treatise, this fact has been kept constantly in view. All unnecessary discussions and scholia have been avoided; and such methods have been adopted as experience and attentive observation, combined with repeated trials, have shown to be most readily comprehended. No attempt has been made to render more intelligible the simple notions of position, magnitude, and direction, which every child derives from observation; but it is believed that these notions have been limited and defined with mathematical precision.

A few symbols, which stand for words and not for operations, have been used, but these are of so great utility in giving *style* and *perspicuity* to the demonstrations that no apology seems necessary for their introduction.

Great pains have been taken to make the page attractive. The figures are large and distinct, and are placed in the middle of the page, so that they fall directly under the eye in immediate connection with the corresponding text. The *given* lines of the figures are full lines, the lines employed as *aids* in the demonstrations are short-dotted, and the *resulting* lines are long-dotted.

In each proposition a concise statement of what is given is printed in one kind of type, of what is required in another, and the demonstration in still another. The reason for each step is indicated in small type between that step and the one following, thus preventing the necessity of interrupting the process of the argument by referring to a previous section. The number of the section, however, on which the reason depends is placed at the side of the page. The constituent parts of the propositions are carefully marked. *Moreover, each distinct assertion in the demonstrations, and each particular direction in the constructions of the figures, begins a new line; and in no case is it necessary to turn the page in reading a demonstration.*

This arrangement presents obvious advantages. The pupil perceives at once what is given and what is required, readily refers to the figure at every step, becomes perfectly familiar with the language of Geometry, acquires facility in simple and accurate expression, rapidly *learns to reason,* and lays a foundation for the complete establishing of the science.

A few propositions have been given that might properly be considered as corollaries. The reason for this is the great difficulty of convincing the average student that any importance should be attached to a corollary. Original exercises, however, have been given, not too numerous or too difficult to discourage the beginner, but well adapted to afford an effectual test of the degree in which he is *mastering* the subjects of his reading. Some of these exercises have been placed in the early part of the work in order that the student may discover, at the outset, that to commit to memory a number of theorems and to reproduce them in an examination is a useless and pernicious labor; but to learn their uses and applications, and to acquire a readiness in exemplifying their utility, is to derive the full benefit of that mathematical training which looks not so much to the *attainment of information* as to the *discipline of the mental faculties.*

It only remains to express my sense of obligation to Dr. D. F. Wells for valuable assistance, and to the University Press for the elegance with which the book has been printed; and also to give assurance that any suggestions relating to the work will be thankfully received.

G. A. WENTWORTH.

Phillips Exeter Academy,
January, 1878.

NOTE TO THIRD EDITION.

In this edition I have endeavored to present a more rigorous, but not less simple, treatment of Parallels, Ratio, and Limits. The changes are not sufficient to prevent the simultaneous use of the old and new editions in the class; still they are very important, and have been made after the most careful and prolonged consideration.

I have to express my thanks for valuable suggestions received from many correspondents; and a special acknowledgment is due from me to Professor C. H. Judson, of Furman University, Greenville, South Carolina, to whom I am indebted for assistance in effecting many improvements in this edition.

TO THE TEACHER.

When the pupil is reading each Book for the first time, it will be well to let him write his proofs on the blackboard in his own language; care being taken that his language be the simplest possible, that the arrangement of work be vertical (without side work), and that the figures be accurately constructed.

This method will furnish a valuable exercise as a language lesson, will cultivate the habit of neat and orderly arrangement of work, and will allow a brief interval for deliberating on each step.

After a Book has been read in this way the pupil should review the Book, and should be required to draw the figures free-hand. He should state and prove the propositions orally, using a pointer to indicate on the figure every line and angle named. He should be encouraged, in reviewing each Book, to do the original exercises; to state the converse of propositions; to determine from the statement, if possible, whether the converse be true or false, and if the converse be true to demonstrate it; and also to give well-considered answers to questions which may be asked him on many propositions.

The Teacher is strongly advised to illustrate, geometrically and arithmetically, the principles of limits. Thus a rectangle with a constant base b, and a variable altitude x, will afford an obvious illustration of the axiomatic truth contained in [4], page 88. If x increase and approach the altitude a as a limit, the area of the rectangle increases and approaches the area of the rectangle $a\,b$ as a limit; if, however, x decrease and approach zero as a limit, the area of the rectangle decreases and approaches zero for a limit. An arithmetical illustration of this truth would be given by multiplying a constant into the approximate values of any repetend. If, for example, we take the constant 60 and the repetend .3333, etc., the approximate values of the repetend will be $\frac{3}{10}$, $\frac{33}{100}$, $\frac{333}{1000}$, $\frac{3333}{10000}$, etc., and these values multiplied by 60 give the series 18, 19.8, 19.98, 19.998, etc., which evidently approach 20 as a limit; but the product of 60 into $\frac{1}{3}$ (the limit of the repetend .333, etc.) is also 20.

Again, if we multiply 60 into the different values of the decreasing series, $\frac{1}{30}$, $\frac{1}{300}$, $\frac{1}{3000}$, $\frac{1}{30000}$, etc., which approaches zero as a limit, we shall get the decreasing series, 2, $\frac{1}{5}$, $\frac{1}{50}$, $\frac{1}{500}$, etc.; and this series evidently approaches zero as a limit.

In this way the pupil may easily be led to a complete comprehension of the whole subject of limits.

The Teacher is likewise advised to give frequent written examinations. These should not be too difficult, and sufficient time should be allowed for accurately constructing the figures, for choosing the best language, and for determining the best arrangement.

The time necessary for the reading of examination-books will be diminished by more than one-half, if the use of the symbols employed in this book be permitted.

G. A. W.

Phillips Exeter Academy,
January, 1879.

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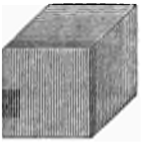
ELEMENTS OF GEOMETRY.

BOOK I.

RECTILINEAR FIGURES.

INTRODUCTORY REMARKS.

A ROUGH block of marble, under the stone-cutter's hammer, may be made to assume *regularity of form.*



If a block be cut in the shape represented in this diagram,

It will have *six flat faces.*

Each *face* of the block is called a *Surface.*

If these surfaces be made smooth by polishing, so that, when a straight-edge is applied to any one of them, the straight-edge in every part will touch the surface, the surfaces are called *Plane Surfaces.*

The sharp edge in which any two of these surfaces meet is called a *Line.*

The place at which any three of these lines meet is called a *Point.*

If now the block be removed, we may think of the *place* occupied by the block as being of precisely the same shape and size as the block itself; also, as having *surfaces* or *boundaries* which separate it from surrounding space. We may likewise think of these surfaces as having *lines* for their boundaries or limits; and of these lines as having *points* for their extremities or limits.

A *Solid,* as the term is used in Geometry, is a limited portion of space.

After we acquire a clear notion of surfaces as boundaries of solids, we can easily conceive of surfaces apart from solids, and suppose them of *unlimited extent.* Likewise we can conceive of lines apart from surfaces, and suppose them of *unlimited length;* of points apart from lines as having *position,* but *no extent.*

DEFINITIONS.

1. DEF. *Space* or *Extension* has three *Dimensions,* called *Length, Breadth,* and *Thickness.*

2. DEF. A *Point* has position without extension.

3. DEF. A *Line* has only *one* of the dimensions of extension, namely, *length.*

The lines which we draw are only imperfect representations of the true lines of Geometry.

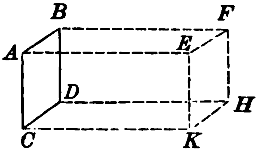
A line may be conceived as traced or generated by a point in motion.

4. DEF. A *Surface* has only *two* of the dimensions of extension, *length* and *breadth.*

A surface may be conceived as generated by a line in motion.

5. DEF. A *Solid* has the *three* dimensions of extension, *length, breadth,* and *thickness.* Hence a solid extends in all directions.

A solid may be conceived as generated by a surface in motion.



Thus, in the diagram, let the upright surface $A\,B\,C\,D$ move to the right to the position $E\,F\,H\,K$. The points A, B, C, and D will generate the lines $A\,E$, $B\,F$, $C\,K$, and $D\,H$ respectively. And the lines $A\,B$, $B\,D$, $D\,C$, and $A\,C$ will generate the surfaces $A\,F$, $B\,H$, $D\,K$, and $A\,K$ respectively. And the surface $A\,B\,C\,D$ will generate the solid $A\,H$.

The relative situation of the two points A and H involves *three, and only three, independent elements.* To pass from A to H it is necessary to move East (if we suppose the direction $A\,E$ to be due East) a distance equal to *A E*, North a distance equal to *E F*, and down a distance equal to *F H*.

These three dimensions we designate for convenience length, breadth, and thickness.

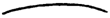
6. The limits (extremities) of lines are points.
The limits (boundaries) of surfaces are lines.
The limits (boundaries) of solids are surfaces.

7. Def. Extension is also called *Magnitude*.

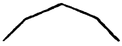
When reference is had to *extent*, lines, surfaces, and solids are called *magnitudes*.

8. Def. A *Straight* line is a line which has the same direction throughout its whole extent.

9. Def. A *Curved* line is a line which changes its direction at every point.



10. Def. A *Broken* line is a series of connected straight lines.



When the word line is used a straight line is meant; and when the word curve is used a curved line is meant.

11. Def. A *Plane Surface*, or a *Plane*, is a surface in which, if any two points be taken, the straight line joining these points will lie wholly in the surface.

12. Def. A *Curved Surface* is a surface no part of which is plane.

13. *Figure* or *form* depends upon the relative position of points. Thus, the figure or form of a line (straight or curved) depends upon the relative position of points in that line; the figure or form of a surface depends upon the relative position of points in that surface.

When reference is had to *form* or *shape*, lines, surfaces, and solids are called *figures*.

14. Def. A *Plane Figure* is a figure, all points of which are in the same plane.

15. Def. *Geometry* is the science which treats of *position, magnitude,* and *form.*

Points, lines, surfaces, and solids, with their relations, are the *geometrical conceptions,* and constitute the subject-matter of Geometry.

16. Plane Geometry treats of plane figures.

Plane figures are either rectilinear, curvilinear, or mixtilinear.

Plane figures formed by straight lines are called *rectilinear* figures; those formed by curved lines are called *curvilinear* figures; and those formed by straight and curved lines are called *mixtilinear* figures.

17. Def. Figures which have the same *form* are called *Similar Figures.* Figures which have the same *extent* are called *Equivalent Figures.* Figures which have the same *form and extent* are called *Equal Figures.*

On Straight Lines.

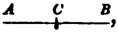
18. If the direction of a straight line and a point in the line be known, the position of the line is known; that is, a straight line is determined in position if its direction and one of its points be known.

Hence, *all straight lines which pass through the same point in the same direction coincide.*

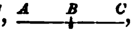
Between two points one, and but one, straight line can be drawn; that is, *a straight line is determined in position if two of its points be known.*

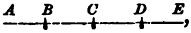
Of all lines between two points, the *shortest* is the straight line; and the straight line is called the *distance* between the two points.

The point *from* which a line is drawn is called its *origin.*

19. If a line, as $C B$, , be produced through C, the portions $C B$ and $C A$ may be regarded as different lines having *opposite directions* from the point C.

Hence, every straight line, as $A B$, , has two opposite directions, namely from A toward B, which is expressed by saying line $A B$, and from B toward A, which is expressed by saying line $B A$.

20. If a straight line change its magnitude, it must become longer or shorter. Thus by prolonging $A B$ to C, , $A C = A B + B C$; and conversely, $B C = A C - A B$.

If a line increase so that it is prolonged by its own magnitude several times in succession, the line is *multiplied*, and the resulting line is called a *multiple* of the given line. Thus, if $A B = B C = C D$, etc., , then $A C = 2 A B$, $A D = 3 A B$, etc.

It must also be possible to divide a given straight line into an assigned number of equal parts. For, assumed that the nth part of a given line were not attainable, then the double, triple, quadruple, of the nth part would not be attainable. Among these multiples, however, we should reach the nth multiple of this nth part, that is, the line itself. Hence, the line itself would not be attainable; which contradicts the hypothesis that we have the given line before us.

Therefore, *it is always possible to add, subtract, multiply, and divide lines of given length.*

21. Since every straight line has the property of direction, it must be true that two straight lines have either the *same direction* or *different directions.*

Two straight lines which have the same direction, without coinciding, can never meet; for if they could meet, then we should have two straight lines passing through the same point in the same direction. Such lines, however, coincide. § 18

22. *Two straight lines which lie in the same plane and have different directions must meet if sufficiently prolonged; and must have one, and but one, point in common.*

Conversely: *Two straight lines lying in the same plane which do not meet have the same direction;* for if they had different directions they would meet, which is contrary to the hypothesis that they do not meet.

Two straight lines which meet have different directions; for if they had the same direction they would never meet (§ 21), which is contrary to the hypothesis that they do meet.

On Plane Angles.

23. Def. An *Angle* is the difference in direction of two lines. The point in which the lines (prolonged if necessary) meet is called the *Vertex*, and the lines are called the *Sides* of the angle.

An angle is designated by placing a letter at its vertex, and one at each of its sides. In reading, we name the three letters, putting the letter at the vertex *between* the other two. When the point is the vertex of but one angle we usually name the letter at the vertex only; thus, in Fig. 1, we read the angle by calling it angle A. But in Fig. 2, H is the common vertex of two angles, so that if we were to say the angle H, it would not be known whether we meant the angle marked 3 or that marked 4. We avoid all ambiguity by reading the former as the angle EHD, and the latter as the angle EHF.

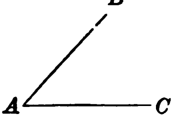


Fig. 1.

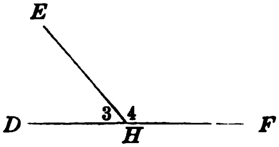
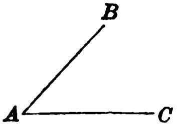
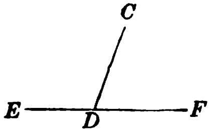


Fig. 2.

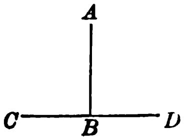
The magnitude of an angle depends wholly upon the *extent of opening* of its sides, and not upon their length. Thus if the sides of the angle *B A C*, namely, *A B* and *A C*, be prolonged, their *extent of opening* will not be altered, and the *size* of the angle, consequently, will not be changed.



24. Def. *Adjacent Angles* are angles having a common vertex and a common side between them. Thus the angles *C D E* and *C D F* are adjacent angles.

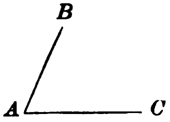


25. Def. A *Right Angle* is an angle included between two straight lines which meet each other so that the two adjacent angles formed by producing one of the lines through the vertex are equal. Thus if the straight line *A B* meet the straight line *C D* so that the adjacent angles *A B C* and *A B D* are equal to one another, each of these angles is called a right angle.

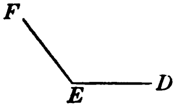


26. Def. *Perpendicular Lines* are lines which make a right angle with each other.

27. Def. An *Acute Angle* is an angle less than a right angle; as the angle *B A C*.

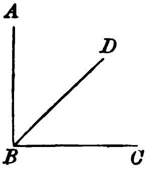


28. Def. An *Obtuse Angle* is an angle greater than a right angle; as the angle *D E F*.

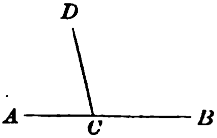


29. Def. Acute and obtuse angles, in distinction from right angles, are called *oblique angles;* and intersecting lines which are not perpendicular to each other are called *oblique lines.*

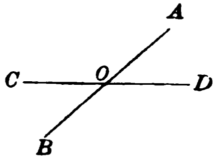
30. Def. The *Complement* of an angle is the difference between a right angle and the given angle. Thus *A B D* is the complement of the angle *D B C*; also *D B C* is the complement of the angle *A B D*.



31. Def. The *Supplement* of an angle is the difference between two right angles and the given angle. Thus $A\,C\,D$ is the supplement of the angle $D\,C\,B$; also $D\,C\,B$ is the supplement of the angle $A\,C\,D$.

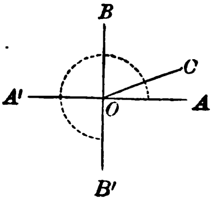


32. Def. *Vertical Angles* are angles which have the same vertex, and their sides extending in opposite directions. Thus the angles $A\,O\,D$ and $C\,O\,B$ are vertical angles, as also the angles $A\,O\,C$ and $D\,O\,B$.



On Angular Magnitude.

33. Let the lines $B\,B'$ and $A\,A'$ be in the same plane, and let $B\,B'$ be perpendicular to $A\,A'$ at the point O.



Suppose the straight line $O\,C$ to move in this plane from coincidence with $O\,A$, about the point O as a pivot, to the position $O\,C$; then the line $O\,C$ *describes* or *generates* the angle $A\,O\,C$.

The *amount of rotation* of the line, from the position $O\,A$ to the position $O\,C$, is the *Angular Magnitude* $A\,O\,C$.

If the rotating line move from the position $O\,A$ to the position $O\,B$, perpendicular to $O\,A$, it generates a right angle; to the position $O\,A'$ it generates two right angles; to the position $O\,B'$, as indicated by the dotted line, it generates three right angles; and if it continue its rotation to the position $O\,A$, whence it started, it generates four right angles.

Hence the whole angular magnitude about a point in a plane is equal to four right angles, and the angular magnitude about a point on one side of a straight line drawn through that point is equal to two right angles.

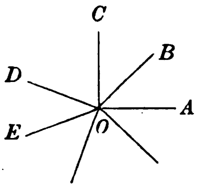


Fig. 1.

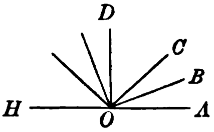
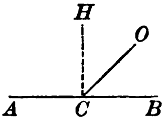


Fig. 2.

34. Now since the angular magnitude about the point O is neither increased nor diminished by the number of lines which radiate from that point, *the sum of all the angles about a point in a plane*, as $AOB + BOC + COD$, etc., in Fig. 1, *is equal to four right angles;* and *the sum of all the angles about a point on one side of a straight line drawn through that point*, as $AOB + BOC + COD$, etc., Fig. 2, *is equal to two right angles.*

Hence two adjacent angles, OCA and OCB, formed by two straight lines, of which one is produced from the point of meeting in both directions, are *supplements* of each other, and may be called *supplementary adjacent* angles.

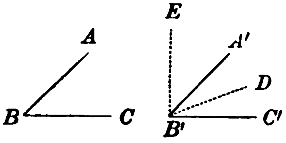


ON THE METHOD OF SUPERPOSITION.

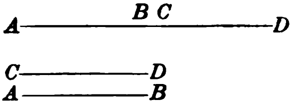
35. The test of the equality of two geometrical magnitudes is that they coincide point for point.

Thus, two straight lines are equal, if they can be so placed that the points at their extremities coincide. Two angles are equal, if they can be so placed that their vertices coincide in position and their sides in direction.

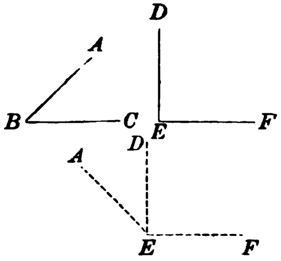
In applying this test of equality, we assume that a line may be moved from one place to another without altering its length; that an angle may be taken up, turned over, and put down, without altering the *difference in direction* of its sides.



This method enables us to compare unequal magnitudes of the same kind. Suppose we have two angles, *A B C* and *A′ B′ C′*. Let the side *B C* be placed on the side *B′ C′*, so that the vertex *B* shall fall on *B′*, then if the side *B A* fall on *B′ A′*, the angle *A B C* equals the angle *A′ B′ C′*; if the side *B A* fall between *B′ C′* and *B′ A′* in the direction *B′ D*, the angle *A B C* is less than *A′ B′ C′*; but if the side *B A* fall in the direction *B′ E*, the angle *A B C* is greater than *A′ B′ C′*.



This method of superposition enables us to add magnitudes of the same kind. Thus, if we have two straight lines *A B* and *C D*, by placing the point *C* on *B*, and keeping *C D* in the same direction with *A B*, we shall have one continuous straight line *A D* equal to the sum of the lines *A B* and *C D*.



Again: if we have the angles *A B C* and *D E F*, by placing the vertex *B* on *E* and the side *B C* in the direction of *E D*, the angle *A B C* will take the position *A E D*, and the angles *D E F* and *A B C* will together equal the angle *A E F*.

Mathematical Terms.

36. Def. A *Demonstration* is a course of reasoning by which the truth or falsity of a particular statement is logically established.

37. Def. A *Theorem* is a truth to be demonstrated.

38. Def. A *Construction* is a graphical representation of a geometrical conception.

39. Def. A *Problem* is a construction to be effected, or a question to be investigated.

40. Def. An *Axiom* is a truth which is admitted without demonstration.

41. Def. A *Postulate* is a problem which is admitted to be possible.

42. Def. A *Proposition* is either a theorem or a problem.

43. Def. A *Corollary* is a truth easily deduced from the proposition to which it is attached.

44. Def. A *Scholium* is a remark upon some particular feature of a proposition.

45. Def. An *Hypothesis* is a supposition made in the enunciation of a proposition, or in the course of a demonstration.

46. AXIOMS.

1. Things which are equal to the same thing are equal to each other.
2. When equals are added to equals the wholes are equal.
3. When equals are taken from equals the remainders are equal.
4. When equals are added to unequals the wholes are unequal.
5. When equals are taken from unequals the remainders are unequal.
6. Things which are double the same thing, or equal things, are equal to each other.
7. Things which are halves of the same thing, or of equal things, are equal to each other.
8. The whole is greater than any of its parts.
9. Every whole is equal to all its parts taken together.

47. POSTULATES.

Let it be granted —

1. That a straight line can be drawn from any one point to any other point.
2. That a straight line can be produced to any distance, or can be terminated at any point.
3. That the circumference of a circle can be described about any centre, at any distance from that centre.

48. SYMBOLS AND ABBREVIATIONS.

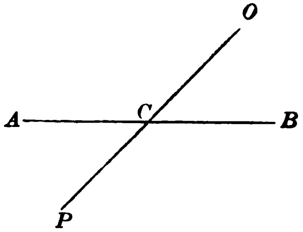
$\therefore$	therefore.
$=$	is (or are) equal to.
$\angle$	angle.
$\angle$s	angles.
$\triangle$	triangle.
$\triangle$s	triangles.
$\parallel$	parallel.
▱	parallelogram
▱s	parallelograms.
$\perp$	perpendicular.
$\perp$s	perpendiculars.
rt. $\angle$	right angle.
rt. $\angle$s	right angles.
$>$	is (or are) greater than.
$<$	is (or are) less than.
rt. $\triangle$	right triangle.
rt. $\triangle$s	right triangles.
$\odot$	circle.
ⓢ	circles.
$+$	increased by.
$-$	diminished by.
$\times$	multiplied by.
$\div$	divided by.

Post.	postulate.
Def.	definition.
Ax.	axiom.
Hyp.	hypothesis.
Cor.	corollary.
Q. E. D.	quod erat demonstrandum.
Q. E. F.	quod erat faciendum.
Adj.	adjacent.
Ext.-int.	exterior-interior.
Alt.-int.	alternate-interior.
Iden.	identical.
Cons.	construction.
Sup.	supplementary.
Sup. adj.	supplementary-adjacent.
Ex.	exercise.
Ill.	illustration.

ON PERPENDICULAR AND OBLIQUE LINES.

PROPOSITION I. THEOREM.

49. *When one straight line crosses another straight line the vertical angles are equal.*



Let line O P cross A B at C.

We are to prove $\angle OCB = \angle ACP$.

$\angle OCA + \angle OCB = 2$ rt. $\angle$s, § 34
(*being sup.-adj.* $\angle$s).

$\angle OCA + \angle ACP = 2$ rt. $\angle$s, § 34
(*being sup.-adj.* $\angle$s).

$\therefore \angle OCA + \angle OCB = \angle OCA + \angle ACP$. Ax. 1.

Take away from each of these equals the common $\angle OCA$.

Then $\angle OCB = \angle ACP$.

In like manner we may prove

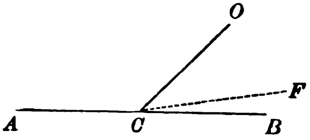
$$\angle ACO = \angle PCB.$$

Q. E. D.

50. COROLLARY. If two straight lines cut one another, the four angles which they make at the point of intersection are together equal to four right angles.

PROPOSITION II. THEOREM.

51. *When the sum of two adjacent angles is equal to two right angles, their exterior sides form one and the same straight line.*



Let the adjacent angles $\angle\, OCA + \angle\, OCB = 2$ **rt.** $\angle s$.

We are to prove AC *and* CB *in the same straight line.*

Suppose CF to be in the same straight line with AC.

Then $\angle\, OCA + \angle\, OCF = 2$ rt. $\angle s$. § 34

(being sup.-adj. $\angle s$*).*

But $\angle\, OCA + \angle\, OCB = 2$ rt. $\angle s$. Hyp.

$\therefore \angle\, OCA + \angle\, OCF = \angle\, OCA + \angle\, OCB.$ Ax. 1.

Take away from each of these equals the common $\angle\, OCA$.

Then $\angle\, OCF = \angle\, OCB.$

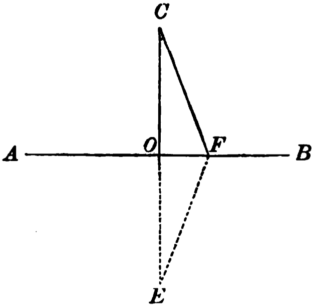
$\therefore$ CB and CF coincide, and cannot form two lines as represented in the figure.

$\therefore$ AC and CB are in the same straight line.

Q. E. D.

PROPOSITION III. THEOREM.

52. *A perpendicular measures the shortest distance from a point to a straight line.*



Let $A B$ be the given straight line, C the given point, and $C O$ the perpendicular.

We are to prove $C O <$ *any other line drawn from C to $A B$, as $C F$.*

Produce $C O$ to E, making $O E = C O$.

Draw $E F$.

On $A B$ as an axis, fold over $O C F$ until it comes into the plane of $O E F$.

The line $O C$ will take the direction of $O E$,

(since $\angle C O F = \angle E O F$, each being a rt. $\angle$).

The point C will fall upon the point E,

(since $O C = O E$ by cons.).

$\therefore$ line $C F =$ line $F E$, § 18

(having their extremities in the same points).

$\therefore C F + F E = 2 C F$,

and $C O + O E = 2 C O$. Cons.

But $C O + O E < C F + F E$, § 18

(a straight line is the shortest distance between two points).

Substitute $2 C O$ for $C O + O E$,

and $2 C F$ for $C F + F E$; then we have

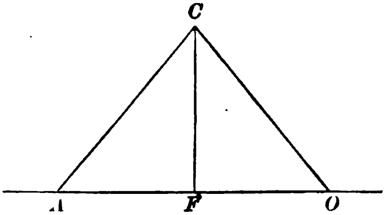
$$2 C O < 2 C F.$$

$$\therefore C O < C F.$$

Q. E. D.

PROPOSITION IV. THEOREM.

53. *Two oblique lines drawn from a point in a perpendicular, cutting off equal distances from the foot of the perpendicular, are equal.*



Let $F C$ be the perpendicular, and $C A$ and $C O$ two oblique lines cutting off equal distances from F.

We are to prove $C A = C O$.

Fold over $C F A$, on $C F$ as an axis, until it comes into the plane of $C F O$.

$F A$ will take the direction of $F O$,
(*since* $\angle C F A = \angle C F O$, *each being a rt.* $\angle$).

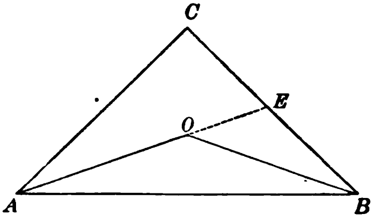
Point A will fall upon point O,
($F A = F O$, *by hyp.*).

$\therefore$ line $C A =$ line $C O$, § 18
(*their extremities being the same points*).

Q. E. D.

PROPOSITION V. THEOREM.

54. *The sum of two lines drawn from a point to the extremities of a straight line is greater than the sum of two other lines similarly drawn, but included by them.*



Let CA and CB be two lines drawn from the point C to the extremities of the straight line AB. Let OA and OB be two lines similarly drawn, but included by CA and CB.

We are to prove $CA + CB > OA + OB$.

Produce AO to meet the line CB at E.

Then $AC + CE > AO + OE$, § 18

(*a straight line is the shortest distance between two points*),

and $BE + OE > BO$. § 18

Add these inequalities, and we have

$$CA + CE + BE + OE > OA + OE + OB.$$

Substitute for $CE + BE$ its equal CB,

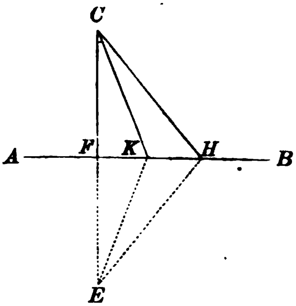
and take away OE from each side of the inequality.

We have $CA + CB > OA + OB$.

Q. E. D.

PROPOSITION VI. THEOREM.

55. *Of two oblique lines drawn from the same point in a perpendicular, cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.*



Let CF be perpendicular to AB, and CK and CH two oblique lines cutting off unequal distances from F.

We are to prove $CH > CK$.

Produce CF to E, *making* $FE = CF$.

Draw EK and EH.

$CH = HE$, and $CK = KE$, § 53

(two oblique lines drawn from the same point in a ⊥, cutting off equal distances from the foot of the ⊥, are equal).

But $CH + HE > CK + KE$, § 54

(The sum of two oblique lines drawn from a point to the extremities of a straight line is greater than the sum of two other lines similarly drawn, but included by them);

$\therefore 2\,CH > 2\,CK$;

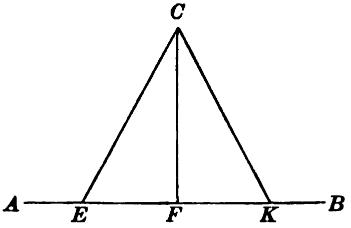
$\therefore CH > CK$.

Q. E. D.

56. COROLLARY. Only two equal straight lines can be drawn from a point to a straight line; and of two unequal lines, the greater cuts off the greater distance from the foot of the perpendicular.

PROPOSITION VII. THEOREM.

57. *Two equal oblique lines, drawn from the same point in a perpendicular, cut off equal distances from the foot of the perpendicular.*



Let CF be the perpendicular, and CE and CK be two equal oblique lines drawn from the point C.

We are to prove $FE = FK$.

Fold over CFA on CF as an axis, until it comes into the plane of CFB.

The line FE will take the direction FK,
($\angle CFE = \angle CFK$, *each being a rt.* $\angle$).

Then the point E must fall upon the point K;

otherwise one of these oblique lines must be more remote from the $\perp$,

and $\therefore$ greater than the other; which is contrary to the hypothesis. § 55

$\therefore FE = FK$.

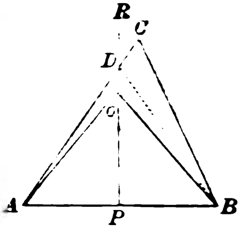
Q. E. D.

PROPOSITION VIII. THEOREM.

58. *If at the middle point of a straight line a perpendicular be erected,*

I. *Any point in the perpendicular is at equal distances from the extremities of the straight line.*

II. *Any point without the perpendicular is at unequal distances from the extremities of the straight line.*



Let PR be a perpendicular erected at the middle of the straight line AB, O any point in PR, and C any point without PR.

I. Draw OA and OB.

We are to prove $OA = OB$.

Since $PA = PB$,

$OA = OB$, § 53

(*two oblique lines drawn from the same point in a ⊥, cutting off equal distances from the foot of the ⊥, are equal*).

II. Draw CA and CB.

We are to prove CA *and* CB *unequal.*

One of these lines, as CA, will intersect the ⊥.

From D, the point of intersection, draw DB.

$$DB = DA,$$ § 53

(two oblique lines drawn from the same point in a ⊥, cutting off equal distances from the foot of the ⊥, are equal).

$$CB < CD + DB,$$ § 18

(a straight line is the shortest distance between two points).

Substitute for DB its equal DA, then

$$CB < CD + DA.$$

But $$CD + DA = CA,$$ Ax. 9.

$$\therefore CB < CA.$$

Q. E. D.

59. The *Locus of a point* is a line, straight or curved, containing all the points which possess a common property.

Thus, the perpendicular erected at the middle of a straight line is the locus of all points equally distant from the extremities of that straight line.

60. SCHOLIUM. Since two points determine the position of a straight line, two points equally distant from the extremities of a straight line determine the perpendicular at the middle point of that line.

Ex. 1. If an angle be a right angle, what is its complement?

2. If an angle be a right angle, what is its supplement?

3. If an angle be $\frac{2}{3}$ of a right angle, what is its complement?

4. If an angle be $\frac{2}{3}$ of a right angle, what is its supplement?

5. Show that the bisectors of two vertical angles form one and the same straight line.

6. Show that the two straight lines which bisect the two pairs of vertical angles are perpendicular to each other.

PROPOSITION IX. THEOREM.

61. *At a point in a straight line only one perpendicular to that line can be drawn; and from a point without a straight line only one perpendicular to that line can be drawn.*

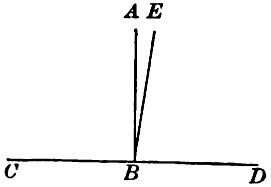


Fig. 1.

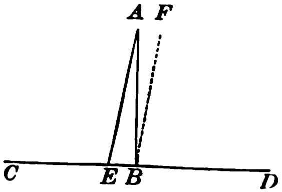


Fig. 2.

Let BA (fig. 1) be perpendicular to CD at the point B.

We are to prove BA the only perpendicular to CD at the point B.

If it be possible, let BE be another line $\perp$ to CD at B.

Then $\angle EBD$ is a rt. $\angle$. § 26

But $\angle ABD$ is a rt. $\angle$. § 26

$\therefore \angle EBD = \angle ABD$. Ax. 1.

That is, a part is equal to the whole; which is impossible.

In like manner it may be shown that no other line but BA is $\perp$ to CD at B.

Let AB (fig. 2) be perpendicular to CD from the point A.

We are to prove AB the only $\perp$ to CD from the point A.

If it be possible, let AE be another line drawn from A $\perp$ to CD.

Conceive $\angle AEB$ to be moved to the right until the vertex E falls on B, the side EB continuing in the line CD.

Then the line EA will take the position BF.

Now if AE be $\perp$ to CD, BF is $\perp$ to CD, and there will be two $\perp$s to CD at the point B; which is impossible.

In like manner, it may be shown that no other line but AB is $\perp$ to CD from A. Q. E. D.

62. COROLLARY. Two lines in the same plane perpendicular to the same straight line have the same direction; otherwise they would meet (§ 22), and we should have two perpendicular lines drawn from their point of meeting to the same line; which is impossible.

ON PARALLEL LINES.

63. *Parallel Lines* are straight lines which lie in the same plane and have the same direction, or opposite directions.

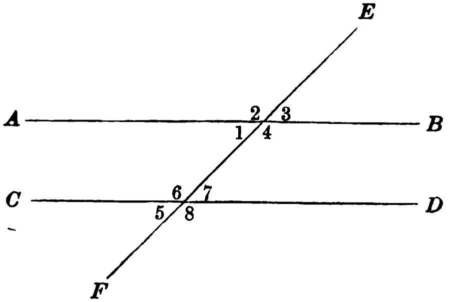
Parallel lines lie in the same direction, when they are on the same side of the straight line joining their origins.

Parallel lines lie in opposite directions, when they are on opposite sides of the straight line joining their origins.

64. *Two parallel lines cannot meet.* § 21

65. *Two lines in the same plane perpendicular to a given line have the same direction* (§ 62), *and are therefore parallel.*

66. *Through a given point only one line can be drawn parallel to a given line.* § 18



If a straight line EF cut two other straight lines AB and CD, it makes with those lines eight angles, to which particular names are given.

The angles 1, 4, 6, 7 are called *Interior* angles.

The angles 2, 3, 5, 8 are called *Exterior* angles.

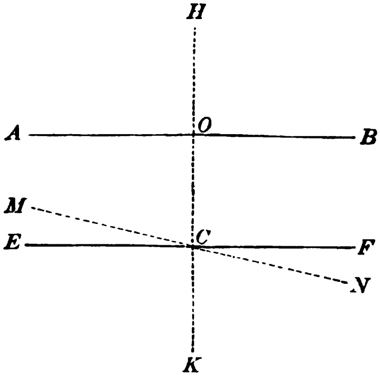
The pairs of angles 1 and 7, 4 and 6 are called *Alternate-interior* angles.

The pairs of angles 2 and 8, 3 and 5 are called *Alternate-exterior* angles.

The pairs of angles 1 and 5, 2 and 6, 4 and 8, 3 and 7 are called *Exterior-interior* angles.

PROPOSITION X. THEOREM.

67. *If a straight line be perpendicular to one of two parallel lines, it is perpendicular to the other.*



Let $A\,B$ and $E\,F$ be two parallel lines, and let $H\,K$ be perpendicular to $A\,B$.

We are to prove $H\,K \perp$ *to* $E\,F$.

Through C draw $M\,N \perp$ to $H\,K$.

Then $M\,N$ is $\parallel$ to $A\,B$. § 65

(*Two lines in the same plane $\perp$ to a given line are parallel*).

But $E\,F$ is $\parallel$ to $A\,B$, Hyp.

$\therefore$ $E\,F$ coincides with $M\,N$. § 66

(*Through the same point only one line can be drawn $\parallel$ to a given line*).

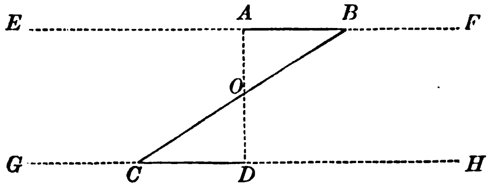
$\therefore$ $E\,F$ is $\perp$ to $H\,K$,

that is $H\,K$ is $\perp$ to $E\,F$.

Q. E. D.

PROPOSITION XI. THEOREM.

68. *If two parallel straight lines be cut by a third straight line the alternate-interior angles are equal.*



Let EF and GH be two parallel straight lines cut by the line BC.

We are to prove $\angle B = \angle C$.

Through O, the middle point of BC, draw $AD \perp$ to GH.

Then AD is likewise $\perp$ to EF, § 67

(a straight line $\perp$ to one of two ||s is $\perp$ to the other),

that is, CD and BA are both $\perp$ to AD.

Apply figure COD to figure BOA so that OD shall fall on OA.

Then OC will fall on OB,

(since $\angle COD = \angle BOA$, being vertical $\angle$s);

and point C will fall upon B,

(since $OC = OB$ by construction).

Then $\perp CD$ will coincide with $\perp BA$, § 61

(from a point without a straight line only one $\perp$ to that line can be drawn).

$\therefore \angle OCD$ coincides with $\angle OBA$, and is equal to it.

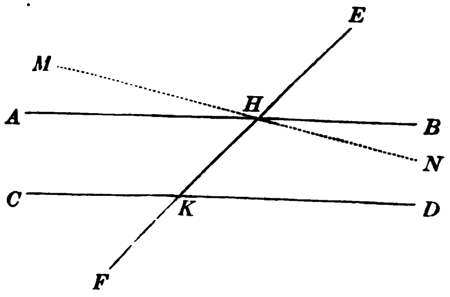
Q. E. D.

SCHOLIUM. By the *converse* of a proposition is meant a proposition which has the hypothesis of the first as conclusion and the conclusion of the first as hypothesis. The converse of a truth is not *necessarily* true. Thus, *parallel lines never meet;* its converse, *lines which never meet are parallel,* is not true unless the lines lie in the same plane.

NOTE. — The converse of many propositions will be omitted, but their statement and demonstration should be required as an important exercise for the student.

PROPOSITION XII. THEOREM.

69. CONVERSELY: *When two straight lines are cut by a third straight line, if the alternate-interior angles be equal, the two straight lines are parallel.*



Let EF cut the straight lines AB and CD in the points H and K, and let the $\angle AHK = \angle HKD$.

We are to prove $AB \parallel$ *to* CD.

Through the point H draw $MN \parallel$ to CD;

then $\angle MHK = \angle HKD$, § 68

(being alt.-int. $\angle$s*).*

But $\angle AHK = \angle HKD$, Hyp.

$\therefore \angle MHK = \angle AHK$. Ax. 1.

$\therefore$ the lines MN and AB coincide.

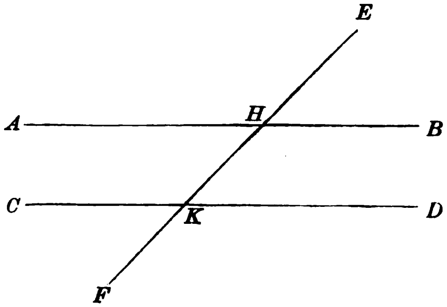
But MN is $\parallel$ to CD; Cons.

$\therefore AB$, which coincides with MN, is $\parallel$ to CD.

Q. E. D.

PROPOSITION XIII. THEOREM.

70. *If two parallel lines be cut by a third straight line, the exterior-interior angles are equal.*



Let AB and CD be two parallel lines cut by the straight line EF, in the points H and K.

We are to prove $\angle EHB = \angle HKD$.

$\angle EHB = \angle AHK$, § 49
(being vertical $\angle$s*).*

But $\angle AHK = \angle HKD$, § 68
(being alt.-int. $\angle$s*).*

$\therefore \angle EHB = \angle HKD$. Ax. 1

In like manner we may prove

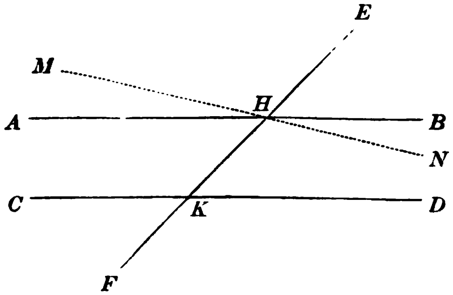
$$\angle EHA = \angle HKC.$$

Q. E. D.

71. Corollary. The alternate-exterior angles, EHB and CKF, and also AHE and DKF, are equal.

PROPOSITION XIV. THEOREM.

72. CONVERSELY: *When two straight lines are cut by a third straight line, if the exterior-interior angles be equal, these two straight lines are parallel.*



Let EF cut the straight lines AB and CD in the points H and K, and let the $\angle EHB = \angle HKD$.

We are to prove $AB \parallel$ *to* CD.

Through the *point* H draw the straight line $MN \parallel$ to CD.

Then $\angle EHN = \angle HKD$, § 70

(*being ext.-int.* $\angle$s).

But $\angle EHB = \angle HKD$. Hyp.

$\therefore \angle EHB = \angle EHN$. Ax. 1.

$\therefore$ the lines MN and AB coincide.

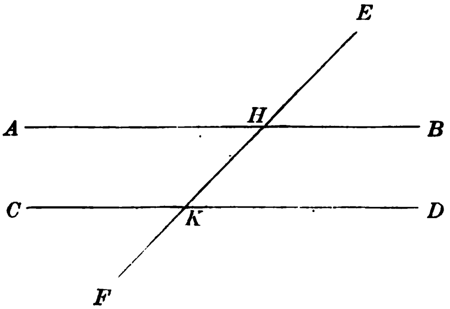
But MN is $\parallel$ to CD, Cons.

$\therefore AB$, which coincides with MN, is $\parallel$ to CD.

Q. E. D.

PROPOSITION XV. THEOREM.

73. *If two parallel lines be cut by a third straight line, the sum of the two interior angles on the same side of the secant line is equal to two right angles.*



Let AB and CD be two parallel lines cut by the straight line EF in the points H and K.

We are to prove $\angle BHK + \angle HKD =$ *two rt.* $\angle$s.

$\angle EHB + \angle BHK = 2$ rt. $\angle$s, § 34

(*being sup.-adj.* $\angle$s).

But $\angle EHB = \angle HKD$, § 70

(*being ext.-int.* $\angle$s).

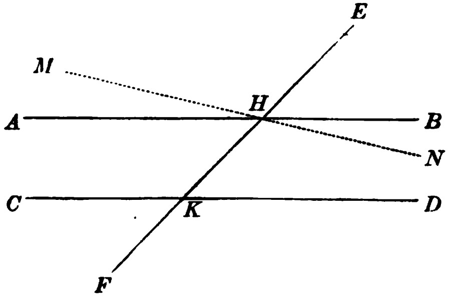
Substitute $\angle HKD$ for $\angle EHB$ in the first equality;

then $\angle BHK + \angle HKD = 2$ rt. $\angle$s.

Q. E. D.

PROPOSITION XVI. THEOREM.

74. CONVERSELY: *When two straight lines are cut by a third straight line, if the two interior angles on the same side of the secant line be together equal to two right angles, then the two straight lines are parallel.*



Let EF cut the straight lines AB and CD in the points H and K, and let the $\angle BHK + \angle HKD$ equal two right angles.

We are to prove $AB \parallel$ *to* CD.

Through the point H draw $MN \parallel$ to CD.

Then $\angle NHK + \angle HKD = 2$ rt. $\measuredangle$s, § 73

(being two interior $\measuredangle$s *on the same side of the secant line).*

But $\angle BHK + \angle HKD = 2$ rt. $\measuredangle$s. Hyp.

$\therefore \angle NHK + \angle HKD = \angle BHK + \angle HKD$. Ax. 1.

Take away from each of these equals the common $\angle HKD$,

then $\angle NHK = \angle BHK$.

$\therefore$ the lines AB and MN coincide.

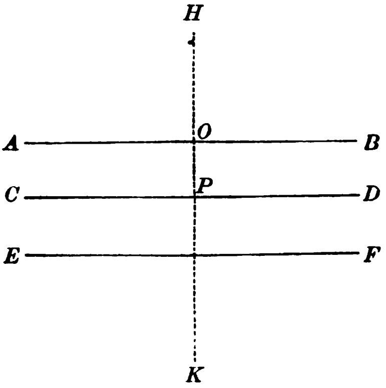
But MN is $\parallel$ to CD; Cons.

$\therefore AB$, which coincides with MN, is $\parallel$ to CD.

Q. E. D.

Proposition XVII. Theorem.

75. *Two straight lines which are parallel to a third straight line are parallel to each other.*



Let AB and CD be parallel to EF.

We are to prove $AB \parallel$ *to* CD.

Draw $HK \perp$ to EF.

Since CD and EF are $\parallel$, HK is $\perp$ to CD, § 67

(*if a straight line be $\perp$ to one of two $\parallel$s, it is $\perp$ to the other also*).

Since AB and EF are $\parallel$, HK is also $\perp$ to AB, § 67

$$\therefore \angle HOB = \angle HPD,$$

(*each being a rt. $\angle$*).

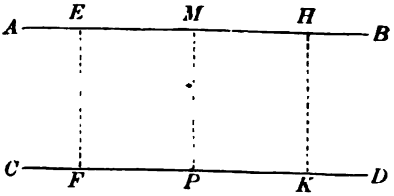
$\therefore AB$ is $\parallel$ to CD, § 72

(*when two straight lines are cut by a third straight line, if the ext.-int. $\angle$s be equal, the two lines are $\parallel$*).

Q. E. D.

Proposition XVIII. Theorem.

76. *Two parallel lines are everywhere equally distant from each other.*



Let AB and CD be two parallel lines, and from any two points in AB, as E and H, let EF and HK be drawn perpendicular to AB.

We are to prove $EF = HK$.

Now EF and HK are $\perp$ to CD, § 67

(a line $\perp$ to one of two $\parallel$s is $\perp$ to the other also).

Let M be the middle point of EH.

Draw $MP \perp$ to AB.

On MP as an axis, fold over the portion of the figure on the right of MP until it comes into the plane of the figure on the left.

MB will fall on MA,

(for $\angle PMH = \angle PME$, each being a rt. $\angle$);

the point H will fall on E,

(for $MH = ME$, by hyp.);

HK will fall on EF,

(for $\angle MHK = \angle MEF$, each being a rt. $\angle$);

and the point K will fall on EF, or EF produced.

Also, PD will fall on PC,

($\angle MPK = \angle MPF$, each being a rt. $\angle$);

and the point K will fall on PC.

Since the point K falls in both the lines EF and PC, it must fall at their point of intersection F.

$\therefore HK = EF$, § 18

(their extremities being the same points).

Q. E. D.

PROPOSITION XIX. THEOREM.

77. *Two angles whose sides are parallel, two and two, and lie in the same direction, or opposite directions, from their vertices, are equal.*

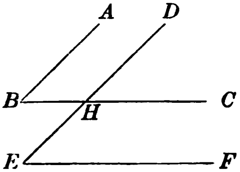


Fig. 1.

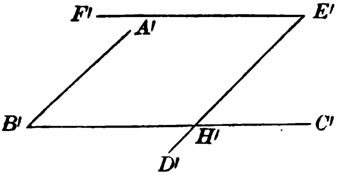


Fig. 2.

Let ∠s B and E (Fig. 1) have their sides BA and ED, and BC and EF respectively, parallel and lying in the same direction from their vertices.

We are to prove the $\angle B = \angle E$.

Produce (if necessary) two sides which are not ∥ until they intersect, as at H;

then $\angle B = \angle DHC$, § 70
(*being ext.-int.* ∠s),

and $\angle E = \angle DHC$, § 70

$\therefore \angle B = \angle E$. Ax. 1

Let ∠s B' and E' (Fig. 2) have $B'A'$ and $E'D'$, and $B'C'$ and $E'F'$ respectively, parallel and lying in opposite directions from their vertices.

We are to prove the $\angle B' = \angle E'$.

Produce (if necessary) two sides which are not ∥ until they intersect, as at H'.

Then $\angle B' = \angle E'H'C'$, § 70
(*being ext.-int.* ∠s),

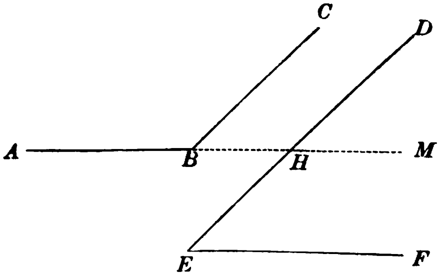
and $\angle E' = \angle E'H'C'$, § 68
(*being alt.-int.* ∠s);

$\therefore \angle B' = \angle E'$, Ax. 1.

Q. E. D.

PROPOSITION XX. THEOREM.

78. *If two angles have two sides parallel and lying in the same direction from their vertices, while the other two sides are parallel and lie in opposite directions, then the two angles are supplements of each other.*



Let $A B C$ and $D E F$ be two angles having $B C$ and $E D$ parallel and lying in the same direction from their vertices, while $E F$ and $B A$ are parallel and lie in opposite directions.

We are to prove $\angle A B C$ *and* $\angle D E F$ *supplements of each other.*

Produce (if necessary) two sides which are not ∥ until they intersect as at H.

$\angle A B C = \angle B H D$, § 70
(*being ext.-int.* $\angle$s).

$\angle D E F = \angle B H E$, § 68
(*being alt.-int.* $\angle$s).

But $\angle B H D$ and $\angle B H E$ are supplements of each other, § 34
(*being sup.-adj.* $\angle$s).

$\therefore \angle A B C$ and $\angle D E F$, the equals of $\angle B H D$ and $\angle B H E$, are supplements of each other.

Q. E. D.

ON TRIANGLES.

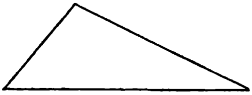
79. DEF. A *Triangle* is a plane figure bounded by three straight lines.

A triangle has six parts, three sides and three angles.

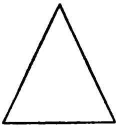
80. When the six parts of one triangle are equal to the six parts of another triangle, each to each, the triangles are said to be *equal in all respects.*

81. DEF. In two equal triangles, the equal angles are called *Homologous* angles, and the equal sides are called *Homologous* sides.

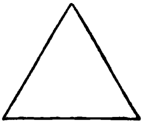
82. In equal triangles the equal sides are opposite the equal angles.



SCALENE.



ISOSCELES.



EQUILATERAL.

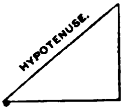
83. DEF. A *Scalene* triangle is one of which no two sides are equal.

84. DEF. An *Isosceles* triangle is one of which two sides are equal.

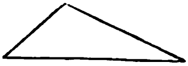
85. DEF. An *Equilateral* triangle is one of which the three sides are equal.

86. DEF. The *Base* of a triangle is the side on which the triangle is supposed to stand.

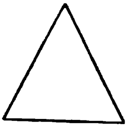
In an isosceles triangle, the side which is not one of the equal sides is considered the base.



RIGHT.



OBTUSE.



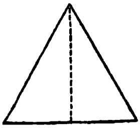
ACUTE.

87. Def. A *Right* triangle is one which has one of the angles a right angle.

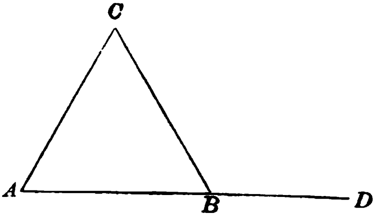
88. Def. The side opposite the right angle is called the *Hypotenuse.*

89. Def. An *Obtuse* triangle is one which has one of the angles an obtuse angle.

90. Def. An *Acute* triangle is one which has all the angles acute.



EQUIANGULAR.



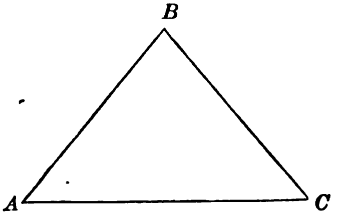
91. Def. An *Equiangular* triangle is one which has all the angles equal.

92. Def. In any triangle, the angle opposite the base is called the *Vertical* angle, and its vertex is called the *Vertex* of the triangle.

93. Def. The *Altitude* of a triangle is the perpendicular distance from the vertex to the base, or the base produced.

94. Def. The *Exterior* angle of a triangle is the angle included between a side and an adjacent side produced, as $\angle CBD$.

95. Def. The two angles of a triangle which are opposite the exterior angle, are called the two *opposite interior* angles, as $\angle$s A and C.



96. *Any side of a triangle is less than the sum of the other two sides.*

Since a straight line is the shortest distance between two points,

$$AC < AB + BC.$$

97. *Any side of a triangle is greater than the difference of the other two sides.*

In the inequality $AC < AB + BC$,

take away AB from each side of the inequality.

Then $$AC - AB < BC; \text{ or}$$

$$BC > AC - AB.$$

Ex. 1. Show that the sum of the distances of any point in a triangle from the three angles of the triangle is greater than half the sum of the sides of the triangle.

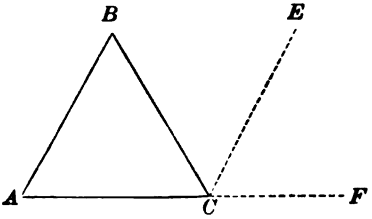
2. Show that the *locus* of all the points at a given distance from a given straight line AB consists of two parallel lines, drawn on opposite sides of AB, and at the given distance from it.

3. Show that the two equal straight lines drawn from a point to a straight line make equal acute angles with that line.

4. Show that, if two angles have their sides perpendicular, each to each, they are either equal or supplementary.

PROPOSITION XXI. THEOREM.

98. *The sum of the three angles of a triangle is equal to two right angles.*



Let ABC be a triangle.

We are to prove $\angle B + \angle BCA + \angle A =$ *two rt.* $\angle$*s.*

Draw $CE \parallel$ to AB, and prolong AC.

Then $\angle ECF + \angle ECB + \angle BCA = 2$ rt. $\angle$s, § 34

(the sum of all the $\angle$*s about a point on the same side of a straight line* $= 2$ *rt.* $\angle$*s).*

But $\angle A = \angle ECF$, § 70

(being ext.-int. $\angle$*s),*

and $\angle B = \angle BCE$, § 68

(being alt.-int. $\angle$*s).*

Substitute for $\angle ECF$ and $\angle BCE$ their equal $\angle$s, A and B.

Then $\angle A + \angle B + \angle BCA = 2$ rt. $\angle$s.

Q. E. D.

99. COROLLARY 1. If the sum of two angles of a triangle be known, the third angle can be found by taking this sum from two right angles.

100. COR. 2. If two triangles have two angles of the one equal to two angles of the other, the third angles will be equal.

101. Cor. 3. If two right triangles have an acute angle of the one equal to an acute angle of the other, the other acute angles will be equal.

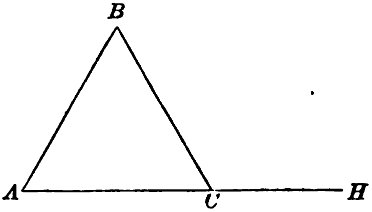
102. Cor. 4. In a triangle there can be but one right angle, or one obtuse angle.

103. Cor. 5. In a right triangle the two acute angles are complements of each other.

104. Cor. 6. In an equiangular triangle, each angle is one third of two right angles, or two thirds of one right angle.

Proposition XXII. Theorem.

105. *The exterior angle of a triangle is equal to the sum of the two opposite interior angles.*



Let BCH be an exterior angle of the triangle ABC.

We are to prove $\angle BCH = \angle A + \angle B$.

$\angle BCH + \angle ACB = 2$ rt. ∠s, § 34
(being sup.-adj. ∠s).

$\angle A + \angle B + \angle ACB = 2$ rt. ∠s, § 98
(three ∠s of a △ = two rt. ∠s).

$\therefore \angle BCH + \angle ACB = \angle A + \angle B + \angle ACB$. Ax. 1.

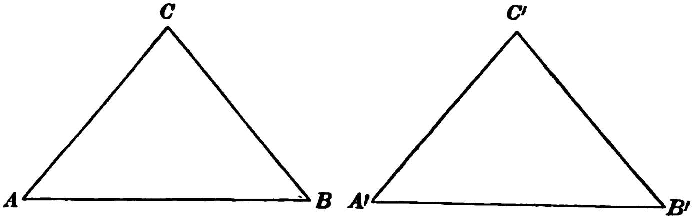
Take away from each of these equals the common $\angle ACB$;

then $\angle BCH = \angle A + \angle B$.

Q. E. D.

PROPOSITION XXIII. THEOREM.

106. *Two triangles are equal in all respects when two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.*



In the triangles $A B C$ and $A' B' C'$, let $A B = A' B'$, $A C = A' C'$, $\angle A = \angle A'$.

We are to prove $\triangle A B C = \triangle A' B' C'$.

Take up the $\triangle A B C$ and place it upon the $\triangle A' B' C'$ so that $A B$ shall coincide with $A' B'$.

Then $A C$ will take the direction of $A' C'$,
(for $\angle A = \angle A'$, by hyp.),

the point C will fall upon the point C',
(for $A C = A' C'$, by hyp.);

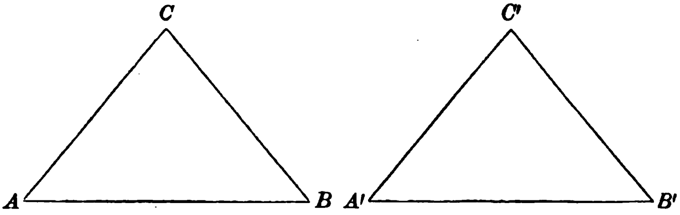
$\therefore C B = C' B'$, § 18
(their extremities being the same points).

$\therefore$ the two $\triangle$s coincide, and are equal in all respects.

Q. E. D.

PROPOSITION XXIV. THEOREM.

107. *Two triangles are equal in all respects when a side and two adjacent angles of the one are equal respectively to a side and two adjacent angles of the other.*



In the triangles ABC and $A'B'C'$, let $AB = A'B'$, $\angle A = \angle A'$, $\angle B = \angle B'$.

We are to prove $\triangle ABC = \triangle A'B'C'$.

Take up $\triangle ABC$ and place it upon $\triangle A'B'C'$, so that AB shall coincide with $A'B'$.

AC will take the direction of $A'C'$,
(for $\angle A = \angle A'$, by hyp.);

the point C, the extremity of AC, will fall upon $A'C'$ or $A'C'$ produced.

BC will take the direction of $B'C'$,
(for $\angle B = \angle B'$, by hyp.);

the point C, the extremity of BC, will fall upon $B'C'$ or $B'C'$ produced.

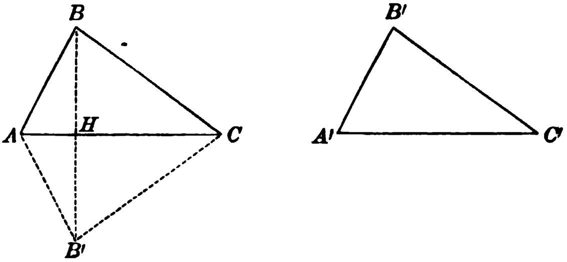
$\therefore$ the point C, falling upon both the lines $A'C'$ and $B'C'$, must fall upon a point common to the two lines, namely, C'.

$\therefore$ the two $\triangle$ coincide, and are equal in all respects.

Q. E. D.

PROPOSITION XXV. THEOREM.

108. *Two triangles are equal when the three sides of the one are equal respectively to the three sides of the other.*



In the triangles $A\,B\,C$ and $A'\,B'\,C'$, let $A\,B = A'\,B'$, $A\,C = A'\,C'$, $B\,C = B'\,C'$.

We are to prove $\triangle\, A\,B\,C = \triangle\, A'\,B'\,C'$.

Place $\triangle\, A'\,B'\,C'$ in the position $A\,B'\,C$, having its greatest side $A'\,C'$ in coincidence with its equal $A\,C$, and its vertex at B', opposite B.

Draw $B\,B'$ intersecting $A\,C$ at H.

Since $A\,B = A\,B'$, Hyp.

point A is at equal distances from B and B'.

Since $B\,C = B'\,C$, Hyp.

point C is at equal distances from B and B'.

$\therefore A\,C$ is $\perp$ to $B\,B'$ at its middle point, § 60

(two points at equal distances from the extremities of a straight line determine the $\perp$ at the middle of that line).

Now if $\triangle\, A\,B'\,C$ be folded over on $A\,C$ as an axis until it comes into the plane of $\triangle\, A\,B\,C$,

$H\,B'$ will fall on $H\,B$,

(for $\angle\, A\,H\,B = \angle\, A\,H\,B'$, each being a rt. $\angle$),

and point B' will fall on B,

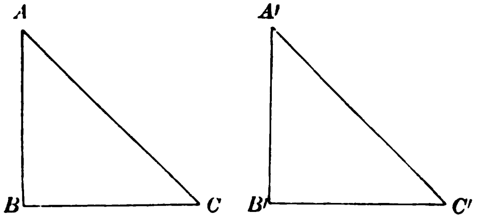
(for $H\,B' = H\,B$).

$\therefore$ the two $\triangle$s coincide, and are equal in all respects.

Q. E. D.

PROPOSITION XXVI. THEOREM.

109. *Two right triangles are equal when a side and the hypotenuse of the one are equal respectively to a side and the hypotenuse of the other.*



In the right triangles $A B C$ **and** $A' B' C'$, **let** $A B = A' B'$, **and** $A C = A' C'$.

We are to prove $\triangle A B C = \triangle A' B' C'$.

Take up the $\triangle A B C$ and place it upon $\triangle A' B' C'$, so that $A B$ will coincide with $A' B'$.

Then $B C$ will fall upon $B' C'$,

(*for* $\angle A B C = \angle A' B' C'$, *each being a rt.* $\angle$),

and point C will fall upon C';

otherwise the equal oblique lines $A C$ and $A' C'$ would cut off *unequal* distances from the foot of the $\perp$, which is impossible, § 57

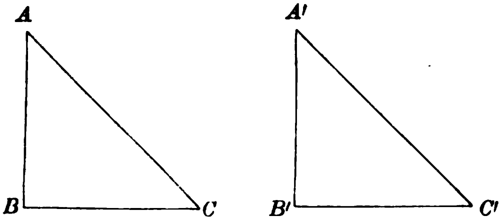
(*two equal oblique lines from a point in a* $\perp$ *cut off equal distances from the foot of the* $\perp$).

$\therefore$ the two $\triangle$s coincide, and are equal in all respects.

Q. E. D.

PROPOSITION XXVII. THEOREM.

110. *Two right triangles are equal when the hypotenuse and an acute angle of the one are equal respectively to the hypotenuse and an acute angle of the other.*



In the right triangles ABC and $A'B'C'$, let $AC = A'C'$, and $\angle A = \angle A'$.

We are to prove $\triangle ABC = \triangle A'B'C'$.

$AC = A'C'$, Hyp.

$\angle A = \angle A'$, Hyp.

then $\angle C = \angle C'$, § 101

(*if two rt. △ have an acute ∠ of the one equal to an acute ∠ of the other, then the other acute ∠ are equal*).

$\therefore \triangle ABC = \triangle A'B'C'$, § 107

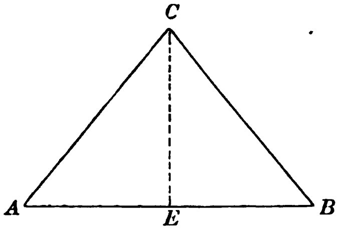
(*two △ are equal when a side and two adj. ∠ of the one are equal respectively to a side and two adj. ∠ of the other*).

Q. E. D.

111. COROLLARY. Two right triangles are equal when a side and an acute angle of the one are equal respectively to an homologous side and acute angle of the other.

PROPOSITION XXVIII. THEOREM.

112. *In an isosceles triangle the angles opposite the equal sides are equal.*



Let ABC be an isosceles triangle, having the sides AC and CB equal.

We are to prove $\angle A = \angle B$.

From C draw the straight line CE so as to bisect the $\angle ACB$.

In the $\triangle$s ACE and BCE,

$AC = BC$, Hyp.

$CE = CE$, Iden.

$\angle ACE = \angle BCE$; Cons.

$\therefore \triangle ACE = \triangle BCE$, § 106

(two $\triangle$s are equal when two sides and the included $\angle$ of the one are equal respectively to two sides and the included $\angle$ of the other).

$\therefore \angle A = \angle B$,

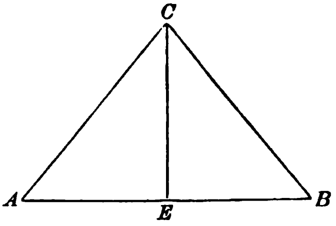
(being homologous $\angle$s of equal $\triangle$s).

Q. E. D.

Ex. If the equal sides of an isosceles triangle be produced, show that the angles formed with the base by the sides produced are equal.

PROPOSITION XXIX. THEOREM.

113. *A straight line which bisects the angle at the vertex of an isosceles triangle divides the triangle into two equal triangles, is perpendicular to the base, and bisects the base.*



Let the line CE bisect the $\angle ACB$ of the isosceles $\triangle ACB$.

We are to prove I. $\triangle ACE = \triangle BCE$;
II. *line* $CE \perp$ *to* AB;
III. $AE = BE$.

I. In the $\triangle$s ACE and BCE,

$AC = BC$, Hyp.

$CE = CE$, Iden.

$\angle ACE = \angle BCE$. Cons.

$\therefore \triangle ACE = \triangle BCE$, § 106

(having two sides and the included $\angle$ of the one equal respectively to two sides and the included $\angle$ of the other).

Also, II. $\angle CEA = \angle CEB$,

(being homologous $\angle$s of equal $\triangle$s).

$\therefore CE$ is $\perp$ to AB,

(a straight line meeting another, making the adjacent $\angle$s equal, is $\perp$ to that line).

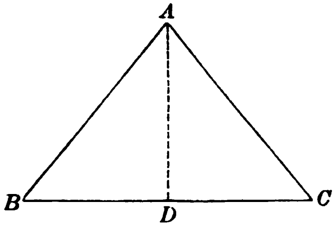
Also, III. $AE = EB$,

(being homologous sides of equal $\triangle$s).

Q. E. D.

PROPOSITION XXX. THEOREM.

114. *If two angles of a triangle be equal, the sides opposite the equal angles are equal, and the triangle is isosceles.*



In the triangle ABC, let the $\angle B = \angle C$.

We are to prove $AB = AC$.

Draw $AD \perp$ to BC.

In the rt. $\triangle$s ADB and ADC,

$AD = AD$, Iden.

$\angle B = \angle C$,

$\therefore$ rt. $\triangle ADB =$ rt. $\triangle ADC$, § 111

(*having a side and an acute $\angle$ of the one equal respectively to a side and an acute $\angle$ of the other*).

$\therefore AB = AC$,

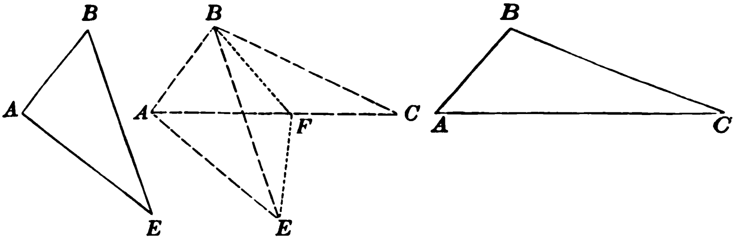
(*being homologous sides of equal $\triangle$s*).

Q. E. D.

Ex. Show that an equiangular triangle is also equilateral.

PROPOSITION XXXI. THEOREM.

115. *If two triangles have two sides of the one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first will be greater than the third side of the second.*



In the $\triangle$s ABC and ABE, let $AB = AB$, $BC = BE$; but $\angle ABC > \angle ABE$.

We are to prove $AC > AE$.

Place the $\triangle$s so that AB of the one shall coincide with AB of the other.

Draw BF so as to bisect $\angle EBC$.

Draw EF.

In the $\triangle$s EBF and CBF

$EB = BC$, Hyp.

$BF = BF$, Iden.

$\angle EBF = \angle CBF$, Cons.

$\therefore$ the $\triangle$s EBF and CBF are equal, § 106

(having two sides and the included $\angle$ of one equal respectively to two sides and the included $\angle$ of the other).

$\therefore EF = FC$,

(being homologous sides of equal $\triangle$s).

Now $AF + FE > AE$, § 96

(the sum of two sides of a $\triangle$ is greater than the third side).

Substitute for FE its equal FC. Then

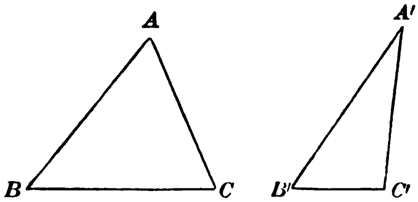
$AF + FC > AE$; or,

$AC > AE$.

Q. E. D.

PROPOSITION XXXII. THEOREM.

116. CONVERSELY: *If two sides of a triangle be equal respectively to two sides of another, but the third side of the first triangle be greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second.*



In the $\triangle\!\!\triangle$ ABC **and** $A'B'C'$, **let** $AB = A'B'$, $AC = A'C'$; **but** $BC > B'C'$.

We are to prove $\angle A > \angle A'$.

If $\angle A = \angle A'$,

then would $\triangle ABC = \triangle A'B'C'$, § 106

(*having two sides and the included* $\angle$ *of the one equal respectively to two sides and the included* $\angle$ *of the other*),

and $BC = B'C'$,

(*being homologous sides of equal* $\triangle\!\!\triangle$).

And if $A < A'$,

then would $BC < B'C'$, § 115

(*if two sides of a* $\triangle$ *be equal respectively to two sides of another* $\triangle$, *but the included* $\angle$ *of the first be greater than the included* $\angle$ *of the second, the third side of the first will be greater than the third side of the second.*)

But both these conclusions are contrary to the hypothesis;

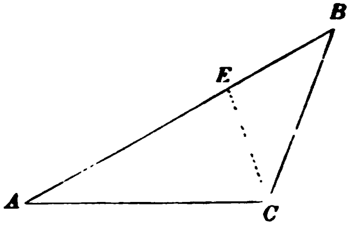
$\therefore \angle A$ does not equal $\angle A'$, and is not less than $\angle A'$.

$$\therefore \angle A > \angle A'.$$

Q. E. D

PROPOSITION XXXIII. THEOREM.

117. *Of two sides of a triangle, that is greater which is opposite the greater angle.*



In the triangle ABC let angle ACB be greater than angle B.

We are to prove $AB > AC$.

Draw CE so as to make $\angle BCE = \angle B$.

Then $EC = EB$, § 112

(being sides opposite equal $\angle$s*).*

Now $AE + EC > AC$, § 96

(the sum of two sides of a $\triangle$ *is greater than the third side).*

Substitute for EC its equal EB. Then

$$AE + EB > AC, \text{ or}$$

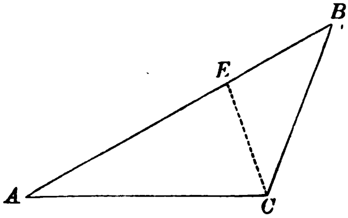
$$AB > AC.$$

Q. E. D.

Ex. ABC and ABD are two triangles on the same base AB, and on the same side of it, the vertex of each triangle being without the other. If AC equal AD, show that BC cannot equal BD.

PROPOSITION XXXIV. THEOREM.

118. *Of two angles of a triangle, that is the greater which is opposite the greater side.*



In the triangle ABC let AB be greater than AC.

We are to prove $\angle ACB > \angle B$.

Take AE equal to AC;

Draw EC.

$\angle AEC = \angle ACE$, § 112

(being ∡ *opposite equal sides).*

But $\angle AEC > \angle B$, § 105

(an exterior ∠ of a △ is greater than either opposite interior ∠),

and $\angle ACB > \angle ACE$.

Substitute for $\angle ACE$ its equal $\angle AEC$, then

$\angle ACB > \angle AEC$.

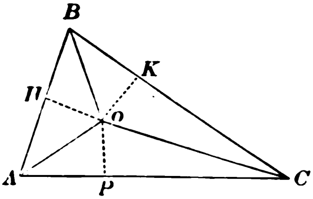
Much more is $\angle ACB > \angle B$.

Q. E. D.

Ex. If the angles ABC and ACB, at the base of an isosceles triangle, be bisected by the straight lines BD, CD, show that DBC will be an isosceles triangle.

PROPOSITION XXXV. THEOREM.

119. *The three bisectors of the three angles of a triangle meet in a point.*



Let the two bisectors of the angles A and C meet at O, and OB be drawn.

We are to prove BO *bisects the* $\angle B$.

Draw the $\perp$s OK, OP, and OH.

In the rt. $\triangle$s OCK and OCP,

$OC = OC$, Iden.

$\angle OCK = \angle OCP$, Cons.

$\therefore \triangle OCK = \triangle OCP$, § 110

(*having the hypotenuse and an acute* $\angle$ *of the one equal respectively to the hypotenuse and an acute* $\angle$ *of the other*).

$\therefore OP = OK$,

(*homologous sides of equal* $\triangle$s).

In the rt. $\triangle$s OAP and OAH,

$OA = OA$, Iden.

$\angle OAP = \angle OAH$, Cons.

$\therefore \triangle OAP = \triangle OAH$, § 110

(*having the hypotenuse and an acute* $\angle$ *of the one equal respectively to the hypotenuse and an acute* $\angle$ *of the other*).

$\therefore OP = OH$,

(*being homologous sides of equal* $\triangle$s).

But we have already shown $OP = OK$,

$\therefore OH = OK$, Ax. 1

Now in rt. $\triangle$s OHB and OKB

$$OH = OK, \text{ and } OB = OB,$$

$$\therefore \triangle OHB = \triangle OKB, \qquad \text{§ 109}$$

(*having the hypotenuse and a side of the one equal respectively to the hypotenuse and a side of the other*),

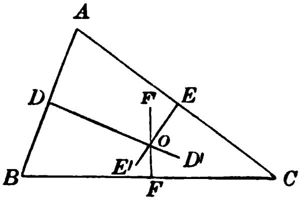
$$\therefore \angle OBH = \angle OBK,$$

(*being homologous* ∠s *of equal* △s).

Q. E. D.

PROPOSITION XXXVI. THEOREM.

120. *The three perpendiculars erected at the middle points of the three sides of a triangle meet in a point.*



Let DD', EE', FF', be three perpendiculars erected at D, E, F, the middle points of AB, AC, and BC.

We are to prove they meet in some point, as O.

The two ⊥s DD' and EE' meet, otherwise they would be parallel, and AB and AC, being ⊥s to these lines from the same point A, would be in the same straight line;

but this is impossible, since they are sides of a △.

Let O be the point at which they meet.

Then, since O is in DD', which is ⊥ to AB at its middle point, it is equally distant from A and B. § 59

Also, since O is in EE', ⊥ to AC at its middle point, it is equally distant from A and C.

$$\therefore O \text{ is equally distant from } B \text{ and } C;$$

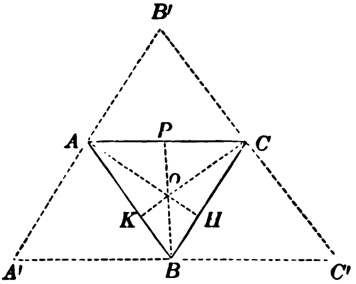
$$\therefore O \text{ is in } FF' \perp \text{ to } BC \text{ at its middle point,} \qquad \text{§ 59}$$

(*the locus of all points equally distant from the extremities of a straight line is the* ⊥ *erected at the middle of that line*).

Q. E. D.

PROPOSITION XXXVII. THEOREM.

121. *The three perpendiculars from the vertices of a triangle to the opposite sides meet in a point.*



In the triangle ABC, let BP, AH, CK, be the perpendiculars from the vertices to the opposite sides.

We are to prove they meet in some point, as O.

Through the vertices A, B, C, draw

$A'B' \parallel$ to BC,

$A'C' \parallel$ to AC,

$B'C' \parallel$ to AB.

In the $\triangle$s ABA' and ABC, we have

$AB = AB$, Iden.

$\angle ABA' = \angle BAC$, § 68

(being alternate interior $\angle$s),

$\angle BAA' = \angle ABC$. § 68

$\therefore \triangle ABA' = \triangle ABC$, § 107

(having a side and two adj. $\angle$s of the one equal respectively to a side and two adj. $\angle$s of the other).

$\therefore A'B = AC$,

(being homologous sides of equal $\triangle$s).

In the $\triangle$s CBC' and ABC,

$BC = BC$, Iden.

$\angle CBC' = \angle BCA$, § 68

(*being alternate interior* $\angle$s).

$\angle BCC' = \angle CBA$. § 68

$\therefore \triangle CBC' = \triangle ABC$, § 107

(*having a side and two adj.* $\angle$s *of the one equal respectively to a side and two adj.* $\angle$s *of the other*).

$\therefore BC' = AC$,

(*being homologous sides of equal* $\triangle$s).

But we have already shown $A'B = AC$,

$\therefore A'B = BC'$, Ax. 1.

$\therefore B$ is the middle point of $A'C'$.

Since BP is $\perp$ to AC, Hyp.

it is $\perp$ to $A'C'$, § 67

(*a straight line which is* $\perp$ *to one of two* $\parallel$s *is* $\perp$ *to the other also*).

But B is the middle point of $A'C'$;

$\therefore BP$ is $\perp$ to $A'C'$ at its middle point.

In like manner we may prove that

AH is $\perp$ to $A'B'$ at its middle point,

and $CK \perp$ to $B'C'$ at its middle point.

$\therefore BP$, AH, and CK are $\perp$s erected at the middle points of the sides of the $\triangle A'B'C'$.

$\therefore$ these $\perp$s meet in a point. § 120

(*the three* $\perp$s *erected at the middle points of the sides of a* $\triangle$ *meet in a point*).

Q. E. D.

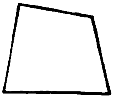
ON QUADRILATERALS.

122. **Def.** A *Quadrilateral* is a plane figure bounded by four straight lines.

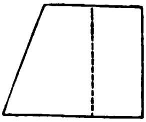
123. **Def.** A *Trapezium* is a quadrilateral which has no two sides parallel.

124. **Def.** A *Trapezoid* is a quadrilateral which has two sides parallel.

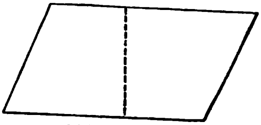
125. **Def.** A *Parallelogram* is a quadrilateral which has its opposite sides parallel.



TRAPEZIUM.



TRAPEZOID.



PARALLELOGRAM.

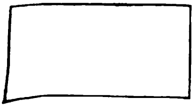
126. **Def.** A *Rectangle* is a parallelogram which has its angles right angles.

127. **Def.** A *Square* is a parallelogram which has its angles right angles, and its sides equal.

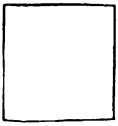
128. **Def.** A *Rhombus* is a parallelogram which has its sides equal, but its angles oblique angles.

129. **Def.** A *Rhomboid* is a parallelogram which has its angles oblique angles.

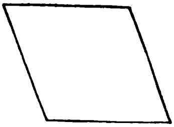
The figure marked parallelogram is also a rhomboid.



RECTANGLE.



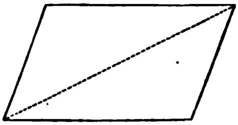
SQUARE.



RHOMBUS.

130. Def. The side upon which a parallelogram stands, and the opposite side, are called its lower and upper *bases;* and the parallel sides of a trapezoid are called its bases.

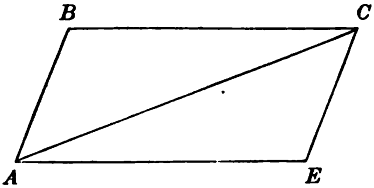
131. Def. The *Altitude* of a parallelogram or trapezoid is the perpendicular distance between its bases.



132. Def. The *Diagonal* of a quadrilateral is a straight line joining any two opposite vertices.

Proposition XXXVIII. Theorem.

133. *The diagonal of a parallelogram divides the figure into two equal triangles.*



Let $ABCE$ be a parallelogram, and AC its diagonal.

We are to prove $\triangle ABC = \triangle AEC$.

In the $\triangle$s ABC and AEC

$AC = AC$, Iden.

$\angle ACB = \angle CAE$, § 68

(being alt.-int. $\angle$s).

$\angle CAB = \angle ACE$, § 68

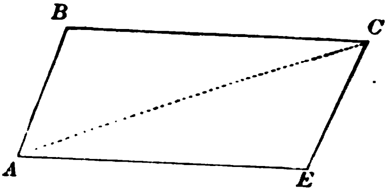
$\therefore \triangle ABC = \triangle AEC$, § 107

(having a side and two adj. $\angle$s of the one equal respectively to a side and two adj. $\angle$s of the other).

Q. E. D.

PROPOSITION XXXIX. THEOREM.

134. *In a parallelogram the opposite sides are equal, and the opposite angles are equal.*



Let the figure $ABCE$ be a parallelogram.

We are to prove $BC = AE$, *and* $AB = EC$,
also, $\angle B = \angle E$, *and* $\angle BAE = \angle BCE$.

Draw AC.

$$\triangle ABC = \triangle AEC,$$

(the diagonal of a ▱ divides the figure into two equal ▲). § 133

$$\therefore BC = AE,$$

and $AB = CE$,

(being homologous sides of equal ▲).

$$\angle B = \angle E,$$

(being homologous ∠ of equal ▲).

$$\angle BAC = \angle ACE,$$

and $\angle EAC = \angle ACB$,

(being homologous ∠ of equal ▲).

Add these last two equalities, and we have

$$\angle BAC + \angle EAC = \angle ACE + \angle ACB;$$

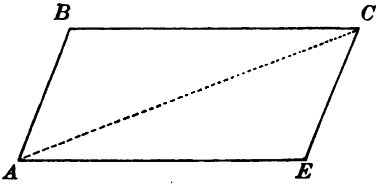
or, $\angle BAE = \angle BCE$.

Q. E. D.

135. COROLLARY. Parallel lines comprehended between parallel lines are equal.

PROPOSITION XL. THEOREM.

136. *If a quadrilateral have two sides equal and parallel, then the other two sides are equal and parallel, and the figure is a parallelogram.*



Let the figure $A\,B\,C\,E$ be a quadrilateral, having the side $A\,E$ equal and parallel to $B\,C$.

We are to prove $A\,B$ *equal and* ∥ *to* $E\,C$.

Draw $A\,C$.

In the $\triangle$s $A\,B\,C$ and $A\,E\,C$

$B\,C = A\,E$, Hyp.

$A\,C = A\,C$, Iden.

$\angle B\,C\,A = \angle C\,A\,E$, § 68

(being alt.-int. $\angle$s*).*

$\therefore \triangle A\,B\,C = \triangle A\,C\,E$, § 106

(having two sides and the included $\angle$ *of the one equal respectively to two sides and the included* $\angle$ *of the other).*

$\therefore A\,B = E\,C$,

(being homologous sides of equal $\triangle$s*).*

Also, $\angle B\,A\,C = \angle A\,C\,E$,

(being homologous $\angle$s *of equal* $\triangle$s*)*;

$\therefore A\,B$ is ∥ to $E\,C$, § 69

(when two straight lines are cut by a third straight line, if the alt.-int. $\angle$s *be equal the lines are parallel).*

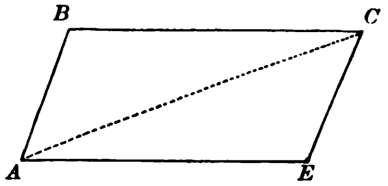
$\therefore$ the figure $A\,B\,C\,E$ is a ▱, § 125

(the opposite sides being parallel).

Q. E. D.

Proposition XLI. Theorem.

137. ***If in a quadrilateral the opposite sides be equal, the figure is a parallelogram.***



Let the figure $ABCE$ be a quadrilateral having $BC = AE$ and $AB = EC$.

We are to prove figure $ABCE$ a ▱.

Draw AC.

In the $\triangle$s ABC and AEC

$BC = AE$, Hyp.

$AB = CE$, Hyp.

$AC = AC$, Iden.

$\therefore \triangle ABC = \triangle AEC$, § 108

(having three sides of the one equal respectively to three sides of the other).

$\therefore \angle ACB = \angle CAE$,

and $\angle BAC = \angle ACE$,

(being homologous ∠s *of equal* $\triangle$s*).*

$\therefore BC$ is ∥ to AE,

and AB is ∥ to EC, § 69

(when two straight lines lying in the same plane are cut by a third straight line, if the alt.-int. ∠s *be equal, the lines are parallel).*

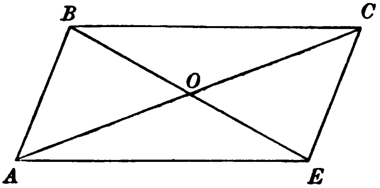
$\therefore$ the figure $ABCE$ is a ▱, § 125

(having its opposite sides parallel).

Q. E. D.

PROPOSITION XLII. THEOREM.

138. *The diagonals of a parallelogram bisect each other.*



Let the figure $ABCE$ be a parallelogram, and let the diagonals AC and BE cut each other at O.

We are to prove $AO = OC$, *and* $BO = OE$.

In the ▲ AOE and BOC

$AE = BC$, § 134

(being opposite sides of a ▱),

$\angle OAE = \angle OCB$, § 68

(being alt.-int. ∠s),

$\angle OEA = \angle OBC$; § 68

$\therefore \triangle AOE = \triangle BOC$, § 107

(having a side and two adj. ∠s of the one equal respectively to a side and two adj. ∠s of the other).

$\therefore AO = OC$,

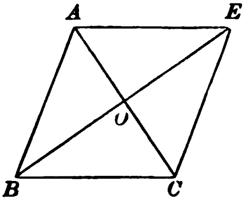
and $BO = OE$.

(being homologous sides of equal ▲).

Q. E. D.

PROPOSITION XLIII. THEOREM.

139. *The diagonals of a rhombus bisect each other at right angles.*



Let the figure $ABCE$ be a rhombus, having the diagonals AC and BE bisecting each other at O.

We are to prove $\angle AOE$ *and* $\angle AOB$ *rt.* $\angle$s.

In the $\triangle$s AOE and AOB,

$AE = AB$, § 128

(being sides of a rhombus);

$OE = OB$, § 138

(the diagonals of a $\square$ *bisect each other)*;

$AO = AO$, Iden.

$\therefore \triangle AOE = \triangle AOB$, § 108

(having three sides of the one equal respectively to three sides of the other);

$\therefore \angle AOE = \angle AOB$,

(being homologous $\angle$s *of equal* $\triangle$s*)*;

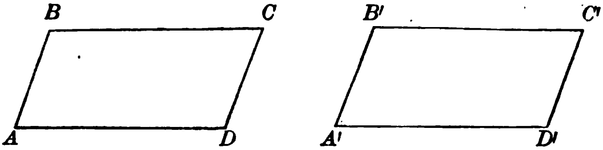
$\therefore \angle AOE$ and $\angle AOB$ are rt. $\angle$s. § 25

(When one straight line meets another straight line so as to make the adj. $\angle$s *equal, each* $\angle$ *is a rt.* $\angle$*).*

Q. E. D.

PROPOSITION XLIV. THEOREM.

140. *Two parallelograms, having two sides and the included angle of the one equal respectively to two sides and the included angle of the other, are equal in all respects.*



In the parallelograms $ABCD$ **and** $A'B'C'D'$, **let** $AB = A'B'$, $AD = A'D'$, **and** $\angle A = \angle A'$.

We are to prove that the ▱s *are equal.*

Apply ▱ $ABCD$ to ▱ $A'B'C'D'$, so that AD will fall on and coincide with $A'D'$.

Then AB will fall on $A'B'$,
(*for* $\angle A = \angle A'$, *by hyp.*),

and the point B will fall on B',
(*for* $AB = A'B'$, *by hyp.*).

Now, BC and $B'C'$ are both ∥ to $A'D'$ and are drawn through point B';

∴ the lines BC and $B'C'$ coincide, § 66

and C falls on $B'C'$ or $B'C'$ produced.

In like manner DC and $D'C'$ are ∥ to $A'B'$ and are drawn through the point D'.

∴ DC and $D'C'$ coincide; § 66

∴ the point C falls on $D'C'$, or $D'C'$ produced;

∴ C falls on both $B'C'$ and $D'C'$;

∴ C must fall on a point common to both, namely, C'.

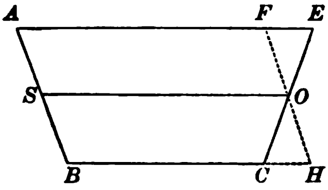
∴ the two ▱s coincide, and are equal in all respects.

Q. E. D.

141. COROLLARY. *Two rectangles having the same base and altitude are equal;* for they may be applied to each other and will coincide.

PROPOSITION XLV. THEOREM.

142. ***The straight line which connects the middle points of the non-parallel sides of a trapezoid is parallel to the parallel sides, and is equal to half their sum.***



Let SO be the straight line joining the middle points of the non-parallel sides of the trapezoid $ABCE$.

We are to prove $SO \parallel$ *to* AE *and* BC;
also $SO = \frac{1}{2}(AE + BC)$.

Through the point O draw $FH \parallel$ to AB,

and produce BC to meet FOH at H.

In the $\triangle$s FOE and COH

$$OE = OC, \qquad \text{Cons.}$$

$$\angle OEF = \angle OCH, \qquad \text{§ 68}$$
(being alt.-int. $\angle$s),

$$\angle FOE = \angle COH, \qquad \text{§ 49}$$
(being vertical $\angle$s).

$$\therefore \triangle FOE = \triangle COH, \qquad \text{§ 107}$$
(having a side and two adj. $\angle$s of the one equal respectively to a side and two adj. $\angle$s of the other).

$$\therefore FE = CH,$$

and $OF = OH,$

(being homologous sides of equal △).

Now $FH = AB,$ § 135

(∥ lines comprehended between ∥ lines are equal);

$$\therefore FO = AS.$$ Ax. 7.

∴ the figure $AFOS$ is a ▱, § 136

(having two opposite sides equal and parallel).

∴ SO is ∥ to AF, § 125

(being opposite sides of a ▱).

SO is also ∥ to BC,

(a straight line ∥ to one of two ∥ lines is ∥ to the other also).

Now $SO = AF,$ § 125

(being opposite sides of a ▱),

and $SO = BH.$ § 125

But $AF = AE - FE,$

and $BH = BC + CH.$

Substitute for AF and BH their equals, $AE - FE$ and $BC + CH$,

and add, observing that $CH = FE$;

then $2\,SO = AE + BC.$

$$\therefore SO = \tfrac{1}{2}(AE + BC).$$

Q. E. D.

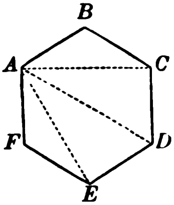
ON POLYGONS IN GENERAL.

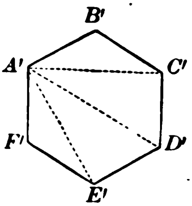
143. DEF. A *Polygon* is a plane figure bounded by straight lines.

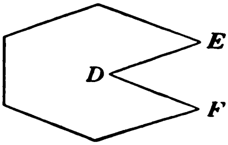
144. DEF. The bounding lines are the ***sides*** of the polygon, and their sum, as $AB + BC + CD$, etc., is the ***Perimeter*** of the polygon.

The angles which the adjacent sides make with each other are the angles of the polygon.

145. DEF. A *Diagonal* of a polygon is a line joining the vertices of two angles not adjacent.







146. DEF. An *Equilateral* polygon is one which has all its sides equal.

147. DEF. An *Equiangular* polygon is one which has all its angles equal.

148. DEF. A *Convex* polygon is one of which no side, when produced, will enter the surface bounded by the perimeter.

149. DEF. Each angle of such a polygon is called a *Salient* angle, and is less than two right angles.

150. DEF. A *Concave* polygon is one of which two or more sides, when produced, will enter the surface bounded by the perimeter.

151. DEF. The angle FDE is called a *Re-entrant* angle.

When the term polygon is used, a *convex* polygon is meant.

The number of sides of a polygon is evidently equal to the number of its angles.

By drawing diagonals from any vertex of a polygon, the figure may be divided into as many triangles as it has sides less two.

152. Def. Two polygons are *Equal*, when they can be divided by diagonals into the same number of triangles, equal each to each, and similarly placed; for the polygons can be applied to each other, and the corresponding triangles will evidently coincide. Therefore the polygons will coincide, and be equal in all respects.

153. Def. Two polygons are *Mutually Equiangular*, if the angles of the one be equal to the angles of the other, each to each, when taken in the same order; as the polygons $ABCDEF$, and $A'B'C'D'E'F'$, in which $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, etc.

154. Def. The equal angles in mutually equiangular polygons are called *Homologous* angles; and the sides which lie *between* equal angles are called *Homologous* sides.

155. Def. Two polygons are *Mutually Equilateral*, if the sides of the one be equal to the sides of the other, each to each, when taken in the same order.

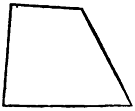


Fig. 1.

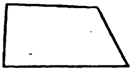


Fig. 2.

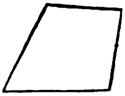


Fig. 3.

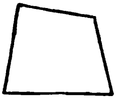


Fig. 4.

Two polygons may be mutually equiangular without being mutually equilateral; as Figs. 1 and 2.

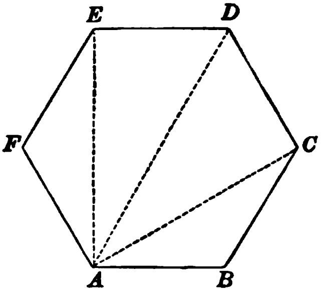
And, *except in the case of triangles*, two polygons may be mutually equilateral without being mutually equiangular; as Figs. 3 and 4.

If two polygons be mutually equilateral and equiangular, *they are equal*, for they may be applied the one to the other so as to coincide.

156. Def. A polygon of three sides is a *Trigon* or *Triangle;* one of four sides is a *Tetragon* or *Quadrilateral;* one of five sides is a *Pentagon;* one of six sides is a *Hexagon;* one of seven sides is a *Heptagon;* one of eight sides is an *Octagon;* one of ten sides is a *Decagon;* one of twelve sides is a *Dodecagon.*

PROPOSITION XLVI. THEOREM.

157. ***The sum of the interior angles of a polygon is equal to two right angles, taken as many times less two as the figure has sides.***



Let the figure *ABCDEF* be a polygon having *n* sides.

We are to prove

$\angle A + \angle B + \angle C$, etc., $= 2$ *rt.* $\angle$s $(n - 2)$.

From the vertex A draw the diagonals AC, AD, and AE.

The sum of the $\angle$s of the $\triangle$s = the sum of the angles of the polygon.

Now there are $(n - 2)$ $\triangle$s,

and the sum of the $\angle$s of each $\triangle = 2$ rt. $\angle$s. § 98

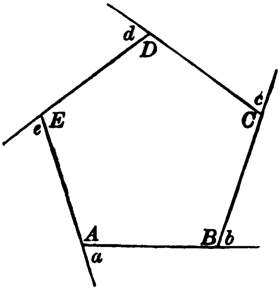
$\therefore$ the sum of the $\angle$s of the $\triangle$s, that is, the sum of the $\angle$s of the polygon $= 2$ rt. $\angle$s $(n - 2)$.

Q. E. D.

158. COROLLARY. The sum of the angles of a quadrilateral equals two right angles taken $(4 - 2)$ times, i. e. equals 4 right angles; and if the angles be all equal, each angle is a right angle. In general, each angle of an equiangular polygon of n sides is equal to $\frac{2(n-2)}{n}$ right angles.

PROPOSITION XLVII. THEOREM.

159. *The exterior angles of a polygon, made by producing each of its sides in succession, are together equal to four right angles.*



Let the figure ABCDE be a polygon, having its sides produced in succession.

We are to prove the sum of the ext. $\angle$s $= 4$ *rt.* $\angle$s.

Denote the int. $\angle$s of the polygon by A, B, C, D, E;

and the ext. $\angle$s by a, b, c, d, e.

$\angle A + \angle a = 2$ rt. $\angle$s, § 34
(*being sup.-adj.* $\angle$s).

$\angle B + \angle b = 2$ rt. $\angle$s. § 34

In like manner each pair of adj. $\angle$s $= 2$ rt. $\angle$s;

$\therefore$ the sum of the interior and exterior $\angle$s $= 2$ rt. $\angle$s taken as many times as the figure has sides,

or, $2\,n$ rt. $\angle$s.

But the interior $\angle$s $= 2$ rt. $\angle$s taken as many times as the figure has sides less two, $= 2$ rt. $\angle$s $(n - 2)$,

or, $2\,n$ rt. $\angle$s $-$ 4 rt. $\angle$s.

$\therefore$ the exterior $\angle$s $= 4$ rt. $\angle$s.

Q. E. D.

EXERCISES.

1. Show that the interior angles of a hexagon are equal to eight right angles.

2. Show that each angle of an equiangular pentagon is $\frac{6}{5}$ of a right angle.

3. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

4. How many sides has the polygon the sum of whose interior angles is equal to the sum of its exterior angles?

5. How many sides has the polygon the sum of whose interior angles is double that of its exterior angles?

6. How many sides has the polygon the sum of whose exterior angles is double that of its interior angles?

7. Every point in the bisector of an angle is equally distant from the sides of the angle; and every point not in the bisector, but within the angle, is unequally distant from the sides of the angle.

8. BAC is a triangle having the angle B double the angle A. If BD bisect the angle B, and meet AC in D, show that BD is equal to AD.

9. If a straight line drawn parallel to the base of a triangle bisect one of the sides, show that it bisects the other also; and that the portion of it intercepted between the two sides is equal to one half the base.

10. $ABCD$ is a parallelogram, E and F the middle points of AD and BC respectively; show that BE and DF will trisect the diagonal AC.

11. If from any point in the base of an isosceles triangle parallels to the equal sides be drawn, show that a parallelogram is formed whose perimeter is equal to the sum of the equal sides of the triangle.

12. If from the diagonal BD of a square $ABCD$, BE be cut off equal to BC, and EF be drawn perpendicular to BD, show that DE is equal to EF, and also to FC.

13. Show that the three lines drawn from the vertices of a triangle to the middle points of the opposite sides meet in a point.

BOOK II.

CIRCLES.

DEFINITIONS.

160. **Def.** A *Circle* is a plane figure bounded by a curved line, all the points of which are equally distant from a point within called the *Centre.*

161. **Def.** The *Circumference* of a circle is the line which bounds the circle.

162. **Def.** A *Radius* of a circle is any straight line drawn from the centre to the circumference, as $O\,A$, Fig. 1.

163. **Def.** A *Diameter* of a circle is any straight line passing through the centre and having its extremities in the circumference, as $A\,B$, Fig. 2.

By the definition of a circle, all its radii are equal. Hence, all its diameters are equal, since the diameter is equal to twice the radius.

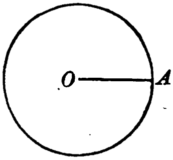


Fig. 1.

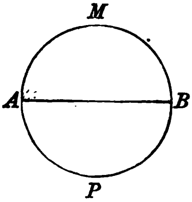


Fig. 2.

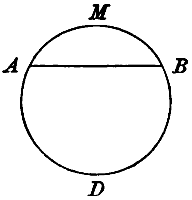


Fig. 3.

164. **Def.** An *Arc* of a circle is any portion of the circumference, as $A\,M\,B$, Fig. 3.

165. **Def.** A *Semi-circumference* is an arc equal to one half the circumference, as $A\,M\,B$, Fig. 2.

166. **Def.** A *Chord* of a circle is any straight line having its extremities in the circumference, as $A\,B$, Fig. 3.

Every chord subtends two arcs whose sum is the circumference. Thus the chord $A\,B$, (Fig. 3), subtends the arc $A\,M\,B$ and the arc $A\,D\,B$. Whenever a chord and its arc are spoken of, the less arc is meant unless it be otherwise stated.

167. **Def.** A *Segment* of a circle is a portion of a circle enclosed by an arc and its chord, as *A M B*, Fig. 1.

168. **Def.** A *Semicircle* is a segment equal to one half the circle, as *A D C*, Fig. 1.

169. **Def.** A *Sector* of a circle is a portion of the circle enclosed by two radii and the arc which they intercept, as *A C B*, Fig. 2.

170. **Def.** A *Tangent* is a straight line which touches the circumference but does not intersect it, however far produced. The point in which the tangent touches the circumference is called the *Point of Contact*, or *Point of Tangency.*

171. **Def.** Two *Circumferences* are tangent to each other when they are tangent to a straight line at the same point.

172. **Def.** A *Secant* is a straight line which intersects the circumference in two points, as *A D*, Fig. 3.

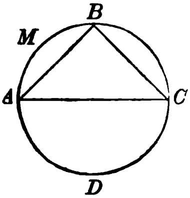


Fig. 1.

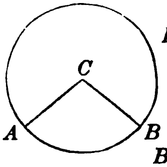


Fig. 2.

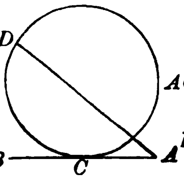


Fig. 3.

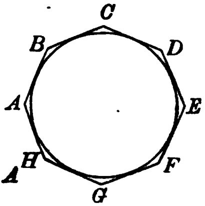


Fig. 4.

173. **Def.** A straight line is *Inscribed* in a circle when its extremities lie in the circumference of the circle, as *A B*, Fig. 1.

An angle is inscribed in a circle when its vertex is in the circumference and its sides are chords of that circumference, as $\angle A B C$, Fig. 1.

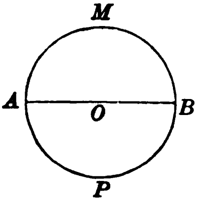
A polygon is inscribed in a circle when its sides are chords of the circle, as $\triangle A B C$, Fig. 1.

A circle is inscribed in a polygon when the circumference touches the sides of the polygon but does not intersect them, as in Fig. 4.

174. **Def.** A polygon is *Circumscribed* about a circle when all the sides of the polygon are tangents to the circle, as in Fig. 4.

A circle is circumscribed about a polygon when the circumference passes through all the vertices of the polygon, as in Fig. 1.

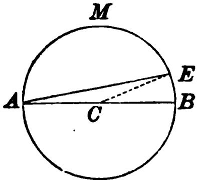
175. Def. *Equal circles* are circles which have equal radii. For if one circle be applied to the other so that their centres coincide their circumferences will coincide, since all the points of both are at the same distance from the centre.



176. *Every diameter bisects the circle and its circumference.* For if we fold over the segment AMB on AB as an axis until it comes into the plane of APB, the arc AMB will coincide with the arc APB; because every point in each is equally distant from the centre O.

Proposition I. Theorem.

177. *The diameter of a circle is greater than any other chord.*



Let AB be the diameter of the circle AMB, and AE any other chord.

We are to prove $AB > AE$.

From C, the centre of the ⊙, draw CE.

$$CE = CB,$$

(being radii of the same circle).

But $$AC + CE > AE, \qquad \S\ 96$$

(the sum of two sides of a △ > the third side).

Substitute for CE, in the above inequality, its equal CB.

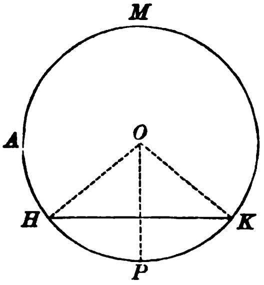
Then $$AC + CB > AE, \text{ or}$$

$$AB > AE.$$

Q. E. D.

PROPOSITION II. THEOREM.

178. ***A straight line cannot intersect the circumference of a circle in more than two points.***



Let *HK* be any line cutting the circumference *AMP*.

We are to prove that HK can intersect the circumference in only two points.

If it be possible, let HK intersect the circumference in three points, H, P, and K.

From O, the centre of the ⊙, draw the radii OH, OP, and OK.

Then OH, OP, and OK are equal, § 163
(being radii of the same circle).

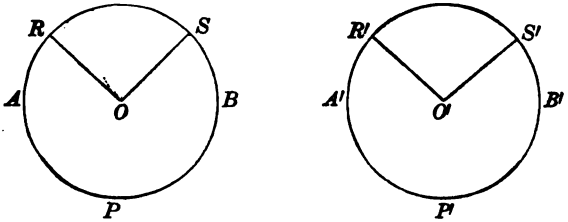
∴ if HK could intersect the circumference in three points, we should have three equal straight lines OH, OP, and OK drawn from the same point to a given straight line, which is impossible, § 58
(only two equal straight lines can be drawn from a point to a straight line).

∴ a straight line can intersect the circumference in only two points.

Q. E. D.

PROPOSITION III. THEOREM.

179. *In the same circle, or equal circles, equal angles at the centre intercept equal arcs on the circumference.*



In the equal circles ABP and A'B'P' let $\angle O = \angle O'$.

We are to prove arc $RS =$ arc $R'S'$.

Apply $\odot ABP$ to $\odot A'B'P'$,

so that $\angle O$ shall coincide with $\angle O'$.

The point R will fall upon R', § 176
(for $OR = O'R'$, being radii of equal ⓢ),

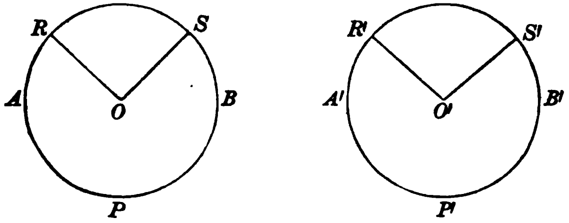
and the point S will fall upon S', § 176
(for $OS = O'S'$, being radii of equal ⓢ).

Then the arc RS must coincide with the arc $R'S'$. For, otherwise, there would be some points in the circumference unequally distant from the centre, which is contrary to the definition of a circle. § 160

Q. E. D.

PROPOSITION IV. THEOREM.

180. CONVERSELY: *In the same circle, or equal circles, equal arcs subtend equal angles at the centre.*



In the equal circles ABP and $A'B'P'$ let arc RS = arc $R'S'$.

We are to prove $\angle ROS = \angle R'O'S'$.

Apply $\odot ABP$ to $\odot A'B'P'$,

so that the radius OR shall fall upon $O'R'$.

Then S, the extremity of arc RS,

will fall upon S', the extremity of arc $R'S'$,
(*for* $RS = R'S'$, *by hyp.*).

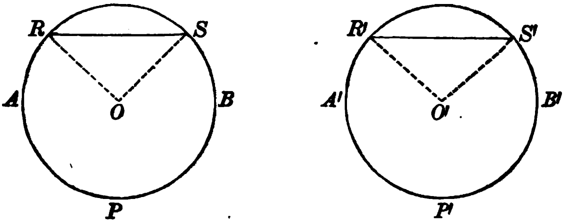
$\therefore OS$ will coincide with $O'S'$, § 18
(*their extremities being the same points*).

$\therefore \angle ROS$ will coincide with, and be equal to, $\angle R'O'S'$.

Q. E. D.

PROPOSITION V. THEOREM.

181. *In the same circle, or equal circles, equal arcs are subtended by equal chords.*



In the equal circles ABP and $A'B'P'$ let arc RS = arc $R'S'$.

We are to prove chord RS = chord $R'S'$.

Draw the radii OR, OS, $O'R'$, and $O'S'$.

In the $\triangle$s ROS and $R'O'S'$

$$OR = O'R',$$ § 176

(being radii of equal Ⓢ),

$$OS = O'S',$$ § 176

$$\angle O = \angle O',$$ § 180

(equal arcs in equal Ⓢ subtend equal $\angle$s at the centre).

$$\therefore \triangle ROS = \triangle R'O'S',$$ § 106

(two sides and the included $\angle$ of the one being equal respectively to two sides and the included $\angle$ of the other).

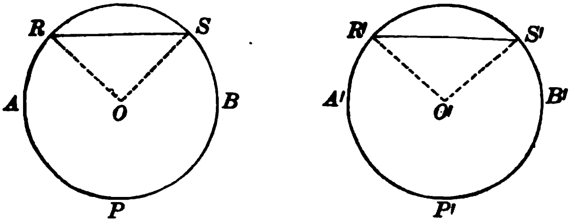
$$\therefore \text{chord } RS = \text{chord } R'S',$$

(being homologous sides of equal $\triangle$s).

Q. E. D.

PROPOSITION VI. THEOREM.

182. CONVERSELY: *In the same circle, or equal circles, equal chords subtend equal arcs.*



In the equal circles ABP and $A'B'P'$, let chord RS = chord $R'S'$.

We are to prove arc RS = arc $R'S'$.

Draw the radii OR, OS, $O'R'$, and $O'S'$.

In the ▲ ROS and $R'O'S'$

$RS = R'S'$, Hyp.

$OR = O'R'$, § 176

(being radii of equal ⊙),

$OS = O'S'$; § 176

$\therefore \triangle ROS = \triangle R'O'S'$, § 108

(three sides of the one being equal to three sides of the other).

$\therefore \angle O = \angle O'$,

(being homologous ∠ of equal ▲).

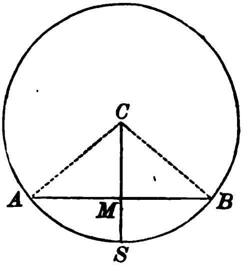
$\therefore$ arc RS = arc $R'S'$, § 179

(in the same ⊙, or equal ⊙, equal ∠ at the centre intercept equal arcs on the circumference).

Q. E. D.

PROPOSITION VII. THEOREM.

183. *The radius perpendicular to a chord bisects the chord and the arc subtended by it.*



Let AB be the chord, and let the radius CS be perpendicular to AB at the point M.

We are to prove $AM = BM$, *and arc* $AS =$ *arc* BS.

Draw CA and CB.

$CA = CB$,

(being radii of the same ⊙);

$\therefore \triangle ACB$ is isosceles, § 84

(the opposite sides being equal);

$\therefore \perp CS$ bisects the base AB and the $\angle C$, § 113

(the ⊥ drawn from the vertex to the base of an isosceles △ bisects the base and the ∠ at the vertex).

$\therefore AM = BM$.

Also, since $\angle ACS = \angle BCS$,

arc $AS =$ arc SB, § 179

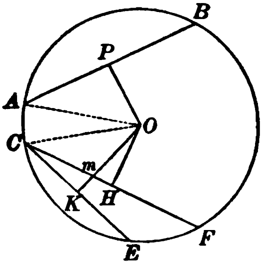
(equal ∡ at the centre intercept equal arcs on the circumference).

Q. E. D.

184. COROLLARY. The perpendicular erected at the middle of a chord passes through the centre of the circle, and bisects the arc of the chord.

PROPOSITION VIII. THEOREM.

185. *In the same circle, or equal circles, equal chords are equally distant from the centre; and of two unequal chords the less is at the greater distance from the centre.*



In the circle $ABEC$ let the chord AB equal the chord CF, and the chord CE be less than the chord CF. Let OP, OH, and OK be $\perp$s drawn to these chords from the centre O.

We are to prove $OP = OH$, *and* $OH < OK$.

Join OA and OC.

In the rt. $\triangle$s AOP and COH

$$OA = OC,$$

(being radii of the same ⊙*)*;

$$AP = CH, \qquad \text{§ 183}$$

(being halves of equal chords);

$$\therefore \triangle AOP = \triangle COH, \qquad \text{§ 109}$$

(two rt. $\triangle$s are equal if they have a side and hypotenuse of the one equal to a side and hypotenuse of the other).

$$\therefore OP = OH,$$

(being homologous sides of equal $\triangle$s).

Again, since $CE < CF$,

the $\perp$ OK will intersect CF in some point, as m.

Now $OK > Om$. Ax. 8

But $Om > OH$, § 52

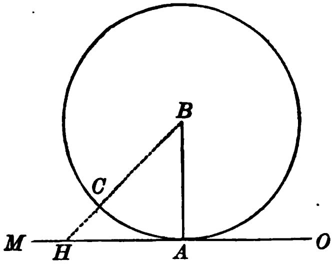
(a $\perp$ is the shortest distance from a point to a straight line).

$\therefore$ much more is $OK > OH$.

Q. E. D.

PROPOSITION IX. THEOREM.

186. *A straight line perpendicular to a radius at its extremity is a tangent to the circle.*



Let BA be the radius, and MO the straight line perpendicular to BA at A.

We are to prove MO *tangent to the circle.*

From B draw any other line to MO, as BCH.

$$BH > BA, \qquad \S\ 52$$

(*a* ⊥ *measures the shortest distance from a point to a straight line*).

∴ point H is without the circumference.

But BH is any other line than BA,

∴ *every* point of the line MO is without the circumference, except A.

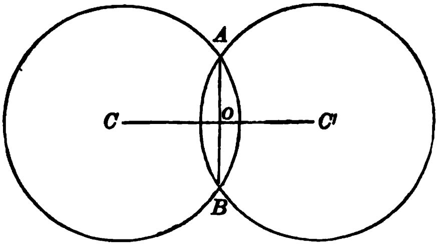
∴ MO is a tangent to the circle at A. § 171

Q. E. D.

187. COROLLARY. When a straight line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact, and therefore a perpendicular to a tangent at the point of contact passes through the centre of the circle.

PROPOSITION X. THEOREM.

188. ***When two circumferences intersect each other, the line which joins their centres is perpendicular to their common chord at its middle point.***



Let C and C' be the centres of two circumferences which intersect each other at A and B, and let the line $C\,C'$ intersect their common chord $A\,B$.

We are to prove $C\,C' \perp$ to $A\,B$ at its middle point.

A ⊥ drawn through the middle of the chord $A\,B$ passes through the centres C and C', § 184

(a ⊥ erected at the middle of a chord passes through the centre of the ⊙).

∴ the line $C\,C'$, having two points in common with this ⊥, must coincide with it.

∴ $C\,C'$ is ⊥ to $A\,B$ at its middle point.

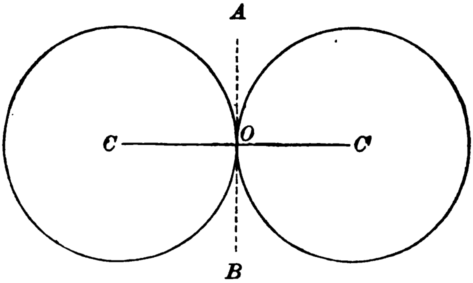
Q. E. D.

Ex. 1. Show that, of all straight lines drawn from a point without a circle to the circumference, the least is that which, when produced, passes through the centre.

Ex. 2. Show that, of all straight lines drawn from a point within or without a circle to the circumference, the greatest is that which meets the circumference after passing through the centre.

PROPOSITION XI. THEOREM.

189. ***When two circumferences are tangent to each other their point of contact is in the straight line joining their centres.***



Let the two circumferences, whose centres are C and C', touch each other at O, in the straight line $A B$, and let $C C'$ be the straight line joining their centres.

We are to prove O *is in the straight line* $C C'$.

A $\perp$ to $A B$, drawn through the point O, passes through the centres C and C', § 187

(*a* $\perp$ *to a tangent at the point of contact passes through the centre of the* $\odot$).

$\therefore$ the line $C C'$, having two points in common with this $\perp$, must coincide with it.

$\therefore$ O is in the straight line $C C'$.

Q. E. D.

Ex. $A B$, a chord of a circle, is the base of an isosceles triangle whose vertex C is without the circle, and whose equal sides meet the circle in D and E. Show that $C D$ is equal to $C E$.

On Measurement.

190. **Def.** To *measure* a quantity of any kind is to find how *many times* it contains another known quantity of the *same kind*. Thus, to measure a line is to find how many times it contains another known line, called the *linear unit.*

191. **Def.** The number which expresses how many times a quantity contains the unit, prefixed to the name of the unit, is called the *numerical measure* of that quantity; as 5 yards, etc.

192. **Def.** Two quantities are *commensurable* if there be some third quantity of the same kind which is contained an exact number of times in each. This third quantity is called the *common measure* of these quantities, and each of the given quantities is called a *multiple* of this common measure.

193. **Def.** Two quantities are *incommensurable* if they have no common measure.

194. **Def.** The magnitude of a quantity is always *relative* to the magnitude of another quantity of the *same kind.* No quantity is great or small except by comparison. This relative magnitude is called their *Ratio,* and this ratio is always an *abstract number.*

When two quantities of the same kind are measured by the *same unit,* their ratio is the ratio of their *numerical measures.*

195. The ratio of a to b is written $\frac{a}{b}$, or $a:b$, and by this is meant:

How many times b is contained in a; a————
or, what part a is of b. b——

I. If b be contained an exact number of times in a their ratio is a *whole number.*

If b be not contained an exact number of times in a, but if there be a common measure which is contained m times in a and n times in b, their ratio is the *fraction* $\frac{m}{n}$.

II. If a and b be incommensurable, their ratio cannot be exactly expressed in figures. But if b be divided into n equal parts, and one of these parts be contained m times in a with a remainder less than $\frac{1}{n}$ part of b, then $\frac{m}{n}$ is an *approximate value* of the ratio $\frac{a}{b}$, correct within $\frac{1}{n}$.

Again, if each of these equal parts of b be divided into n equal parts; that is, if b be divided into n^2 equal parts, and if one of these parts be contained m' times in a with a remainder less than $\frac{1}{n^2}$ part of b, then $\frac{m'}{n^2}$ is a *nearer approximate value* of the ratio $\frac{a}{b}$, correct within $\frac{1}{n^2}$.

By continuing this process, a series of variable values, $\frac{m}{n}$, $\frac{m'}{n^2}$, $\frac{m''}{n^8}$, etc., will be obtained, which will differ less and less from the exact value of $\frac{a}{b}$. We may thus find a fraction which shall differ from this exact value by as little as we please, that is, by less than any assigned quantity.

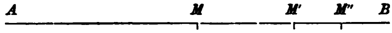
Hence, an *incommensurable ratio* is the *limit* toward which its successive approximate values are constantly tending.

ON THE THEORY OF LIMITS.

196. DEF. When a quantity is regarded as having a *fixed* value, it is called a *Constant;* but, when it is regarded, under the conditions imposed upon it, as having an *indefinite number* of *different* values, it is called a *Variable.*

197. DEF. When it can be shown that the value of a variable, measured at a series of definite intervals, can by indefinite continuation of the series be made to differ from a given constant by less than any assigned quantity, however small, but cannot be made absolutely equal to the constant, that constant is called the *Limit* of the variable, and the variable is said to *approach indefinitely to its limit.*

If the variable be increasing, its limit is called a *superior* limit; if decreasing, an *inferior* limit.

198. Suppose a point 

to move from A toward B, under the conditions that the first second it shall move one-half the distance from A to B, that is, to M; the next second, one-half the remaining distance, that is, to M'; the next second, one-half the remaining distance, that is, to M'', and so on indefinitely.

Then it is evident that the moving point *may approach as near to B as we please, but will never arrive at B.* For, however near it may be to B at any instant, the next second it will pass over one-half the interval still remaining; it must, therefore, approach nearer to B, since *half* the interval still remaining is *some* distance, but will not reach B, since *half* the interval still remaining is not the *whole* distance.

Hence, the distance from A to the moving point is an increasing variable, which indefinitely approaches the constant AB as its *limit;* and the distance from the moving point to B is a decreasing variable, which indefinitely approaches the constant zero as its *limit.*

If the length of AB be two inches, and the variable be denoted by x, and the difference between the variable and its limit, by v:

after one second, $x = 1$, $v = 1$;
after two seconds, $x = 1 + \frac{1}{2}$, $v = \frac{1}{2}$;
after three seconds, $x = 1 + \frac{1}{2} + \frac{1}{4}$, $v = \frac{1}{4}$;
after four seconds, $x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, $v = \frac{1}{8}$;
and so on indefinitely.

Now the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ etc., is evidently less than 2; but by taking a great number of terms, the sum can be made to differ from 2 by as little as we please. Hence 2 is the limit of the sum of the series, when the number of the terms is increased indefinitely; and 0 is the limit of the variable difference between this variable sum and 2.

lim. will be used as an abbreviation for limit.

199. [1] *The difference between a variable and its limit is a variable whose limit is zero.*

[2] *If two or more variables,* v, v', v'', *etc., have zero for a limit, their sum,* $v + v' + v''$, *etc., will have zero for a limit.*

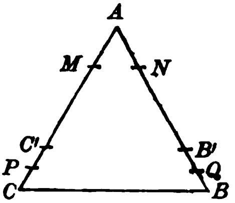
[3] *If the limit of a variable,* v, *be zero, the limit of* $a \pm v$ *will be the constant* a, *and the limit of* $a \times v$ *will be zero.*

[4] *The product of a constant and a variable is also a variable, and the limit of the product of a constant and a variable is the product of the constant and the limit of the variable.*

[5] *The sum or product of two variables, both of which are either increasing or decreasing, is also a variable.*

Proposition I.

[6] *If two variables be always equal, their limits are equal.*



Let the two variables AM and AN be always equal, and let AC and AB be their respective limits.

We are to prove $AC = AB$.

Suppose $AC > AB$. Then we may diminish AC to some value AC' such that $AC' = AB$.

Since AM approaches indefinitely to AC, we may suppose that it has reached a value AP greater than AC'.

Let AQ be the corresponding value of AN.

Then $AP = AQ$.

Now $AC' = AB$.

But both of these equations cannot be true, for $AP > AC'$, and $AQ < AB$. $\therefore AC$ cannot be greater than AB.

Again, suppose $AC < AB$. Then we may diminish AB to some value AB' such that $AC = AB'$.

Since AN approaches indefinitely to AB we may suppose that it has reached a value AQ greater than AB'.

Let AP be the corresponding value of AM.

Then $AP = AQ$.

Now $AC = AB'$.

But both of these equations cannot be true, for $AP < AC$, and $AQ > AB'$. $\therefore AC$ cannot be less than AB.

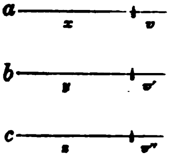
Since AC cannot be greater or less than AB, it must be equal to AB. Q. E. D.

[7] Corollary 1. *If two variables be in a constant ratio, their limits are in the same ratio.* For, let x and y be two variables having the constant ratio r, then $\frac{x}{y} = r$, or, $x = ry$, therefore $lim.\ (x) = lim.\ (ry) = r \times lim.\ (y)$, therefore $\frac{lim.\ (x)}{lim.\ (y)} = r$.

[8] Cor. 2. Since an incommensurable ratio is the limit of its successive approximate values, *two incommensurable ratios* $\frac{a}{b}$ *and* $\frac{a'}{b'}$ *are equal if they always have the same approximate values when expressed within the same measure of precision.*

PROPOSITION II.

[9] *The limit of the algebraic sum of two or more variables is the algebraic sum of their limits.*



Let x, y, z, **be variables,** a, b, **and** c, **their respective limits, and** v, v', **and** v'', **the variable differences between** x, y, z, **and** a, b, c, **respectively.**

We are to prove $lim.\ (x+y+z)=a+b+c$.

Now, $x=a-v$, $y=b-v'$, $z=c-v''$.

Then, $x+y+z=a-v+b-v'+c-v''$.

$\therefore lim.\ (x+y+z)=lim.\ (a-v+b-v'+c-v'')$. [6]

But, $lim.\ (a-v+b-v'+c-v'')=a+b+c$. [3]

$\therefore lim.\ (x+y+z)=a+b+c$.

Q. E. D.

PROPOSITION III.

[10] *The limit of the product of two or more variables is the product of their limits.*

Let x, y, z, **be variables,** a, b, c, **their respective limits, and** v, v', v'', **the variable differences between** x, y, z, **and** a, b, c, **respectively.**

We are to prove $lim.\ (x\,y\,z)=a\,b\,c$.

Now, $x=a-v$, $y=b-v'$, $z=c-v''$.

Multiply these equations together.

Then, $x\,y\,z=a\,b\,c\mp$ terms which contain one or more of the factors v, v', v'', and hence have zero for a limit. [3]

$\therefore lim.\ (x\,y\,z)=lim.\ (a\,b\,c\mp$ terms whose limits are zero). [6]

But $lim.\ (a\,b\,c\mp$ terms whose limits are zero$)=a\,b\,c$.

$\therefore lim.\ (x\,y\,z)=a\,b\,c$.

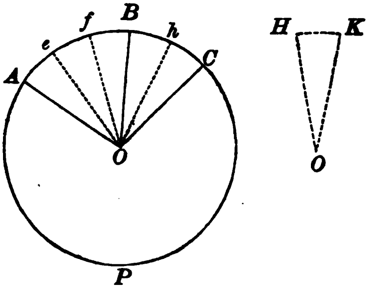
Q. E. D.

For *decreasing* variables the proofs are similar.

NOTE. — In the application of the principles of limits, reference to this section (§ 199) will always include the *fundamental truth* of limits contained in Proposition I.; and it will be left as an exercise for the student to determine in each case what other truths of this section, if any, are included in the reference.

PROPOSITION XII. THEOREM.

200. *In the same circle, or equal circles, two commensurable arcs have the same ratio as the angles which they subtend at the centre.*



In the circle APC let the two arcs be AB and AC, and AOB and AOC the ∠s which they subtend.

We are to prove $\dfrac{\text{arc } AB}{\text{arc } AC} = \dfrac{\angle AOB}{\angle AOC}.$

Let HK be a common measure of AB and AC.

Suppose HK to be contained in AB three times, and in AC five times.

Then $$\frac{\text{arc } AB}{\text{arc } AC} = \frac{3}{5}.$$

At the several points of division on AB and AC draw radii.

These radii will divide $\angle AOC$ into five equal parts, of which $\angle AOB$ will contain three, § 180

(in the same ⊙, or equal ⊙s, equal arcs subtend equal ∠s at the centre).

$$\therefore \frac{\angle AOB}{\angle AOC} = \frac{3}{5}.$$

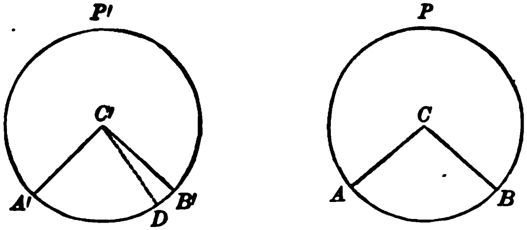
But $$\frac{\text{arc } AB}{\text{arc } AC} = \frac{3}{5}.$$

$$\therefore \frac{\text{arc } AB}{\text{arc } AC} = \frac{\angle AOB}{\angle AOC}.$$ Ax. 1.

Q. E. D.

PROPOSITION XIII. THEOREM.

201. ***In the same circle, or in equal circles, incommensurable arcs have the same ratio as the angles which they subtend at the centre.***



In the two equal ⊛ ABP and $A'B'P'$ let AB and $A'B'$ be two incommensurable arcs, and C, C' the ∡ which they subtend at the centre.

We are to prove $\frac{\text{arc } A'B'}{\text{arc } AB} = \frac{\angle C'}{\angle C}$.

Let AB be divided into any number of equal parts, and let one of these parts be applied to $A'B'$ as often as it will be contained in $A'B'$.

Since AB and $A'B'$ are incommensurable, a certain number of these parts will extend from A' to some point, as D, leaving a remainder DB' less than one of these parts.

Draw $C'D$.

Since AB and $A'D$ are commensurable,

$$\frac{\text{arc } A'D}{\text{arc } AB} = \frac{\angle A'C'D}{\angle ACB}, \qquad \S\ 200$$

(*two commensurable arcs have the same ratio as the ∡ which they subtend at the centre*).

Now suppose the number of parts into which AB is divided to be continually increased; then the length of each part will become less and less, and the point D will approach nearer and nearer to B', that is, the arc $A'D$ will approach the arc $A'B'$ as its limit, and the $\angle A'C'D$ the $\angle A'C'B'$ as its limit.

Then the limit of $\frac{\text{arc } A'D}{\text{arc } AB}$ will be $\frac{\text{arc } A'B'}{\text{arc } AB}$,

and the limit of $\frac{\angle A'C'D}{\angle ACB}$ will be $\frac{\angle A'C'B'}{\angle ACB}$.

Moreover, the corresponding values of the two variables, namely,

$$\frac{\text{arc } A'D}{\text{arc } AB} \text{ and } \frac{\angle A'C'D}{\angle ACB},$$

are equal, however near these variables approach their limits.

$\therefore$ their limits $\frac{\text{arc } A'B'}{\text{arc } AB}$ and $\frac{\angle A'C'B'}{\angle ACB}$ are equal. § 199

Q. E. D.

202. Scholium. *An angle at the centre is said to be measured by its intercepted arc.* This expression means that an angle at the centre is such part of the angular magnitude about that point (four right angles) as its intercepted arc is of the whole circumference.

A circumference is divided into 360 equal arcs, and each arc is called a degree, denoted by the symbol (°).

The angle at the centre which one of these equal arcs subtends is also called a degree.

A quadrant (one-fourth a circumference) contains therefore 90°; and a right angle, subtended by a quadrant, contains 90°.

Hence an angle of 30° is $\frac{1}{3}$ of a right angle, an angle of 45° is $\frac{1}{2}$ of a right angle, an angle of 135° is $\frac{3}{2}$ of a right angle.

Thus we get a definite idea of an angle if we know the number of degrees it contains.

A degree is subdivided into sixty equal parts called minutes, denoted by the symbol (′).

A minute is subdivided into sixty equal parts called seconds, denoted by the symbol (″).

PROPOSITION XIV. THEOREM.

203. ***An inscribed angle is measured by one-half of the arc intercepted between its sides.***

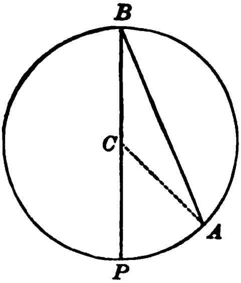


Fig. 1.

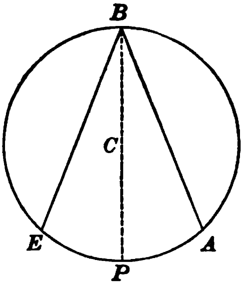


Fig. 2.

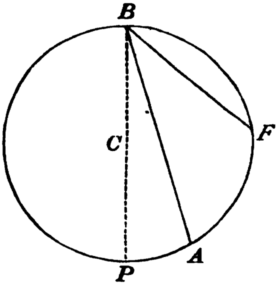


Fig. 3.

CASE I.

In the circle PAB (Fig. 1), let the centre C be in one of the sides of the inscribed angle B.

We are to prove $\angle B$ *is measured by* $\frac{1}{2}$ *arc* PA.

Draw CA.

$CA = CB$,

(being radii of the same ⊙*).*

$\therefore \angle B = \angle A$, § 112

(being opposite equal sides).

$\angle PCA = \angle B + \angle A$. § 105

(the exterior $\angle$ *of a* $\triangle$ *is equal to the sum of the two opposite interior* $\angle s$*).*

Substitute in the above equality $\angle B$ for its equal $\angle A$.

Then we have $\angle PCA = 2 \angle B$.

But $\angle PCA$ is measured by AP, § 202

(the $\angle$ *at the centre is measured by the intercepted arc).*

$\therefore 2 \angle B$ is measured by AP.

$\therefore \angle B$ is measured by $\frac{1}{2}\, AP$.

CASE II.

In the circle BAE (Fig. 2), let the centre C fall within the angle EBA.

We are to prove $\angle EBA$ *is measured by* $\frac{1}{2}$ *arc* EA.

Draw the diameter BCP.

$\angle PBA$ is measured by $\frac{1}{2}$ arc PA, (Case I.)

$\angle PBE$ is measured by $\frac{1}{2}$ arc PE, (Case I.)

$\therefore \angle PBA + \angle PBE$ is measured by $\frac{1}{2}$ (arc PA + arc PE).

$\therefore \angle EBA$ is measured by $\frac{1}{2}$ arc EA.

CASE III.

In the circle BFP (Fig. 3), let the centre C fall without the angle ABF.

We are to prove $\angle ABF$ *is measured by* $\frac{1}{2}$ *arc* AF.

Draw the diameter BCP.

$\angle PBF$ is measured by $\frac{1}{2}$ arc PF, (Case I.)

$\angle PBA$ is measured by $\frac{1}{2}$ arc PA, (Case I.)

$\therefore \angle PBF - \angle PBA$ is measured by $\frac{1}{2}$ (arc PF − arc PA).

$\therefore \angle ABF$ is measured by $\frac{1}{2}$ arc AF.

Q. E. D.

204. COROLLARY 1. An angle inscribed in a semicircle is a right angle, for it is measured by one-half a semi-circumference, or by 90°.

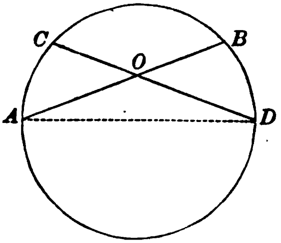
205. COR. 2. An angle inscribed in a segment greater than a semicircle is an acute angle; for it is measured by an arc less than one-half a semi-circumference; i. e. by an arc less than 90°.

206. COR. 3. An angle inscribed in a segment less than a semicircle is an obtuse angle, for it is measured by an arc greater than one-half a semi-circumference; i. e. by an arc greater than 90°.

207. COR. 4. All angles inscribed in the same segment are equal, for they are measured by one-half the same arc.

PROPOSITION XV. THEOREM.

208. ***An angle formed by two chords, and whose vertex lies between the centre and the circumference, is measured by one-half the intercepted arc plus one-half the arc intercepted by its sides produced.***



Let the $\angle\ A\,O\,C$ be formed by the chords $A\,B$ and $C\,D$.

We are to prove

$\angle\ A\ O\ C$ *is measured by* $\frac{1}{2}$ *arc* $A\ C + \frac{1}{2}$ *arc* $B\,D$.

Draw $A\ D$.

$\angle\ C\,O\,A = \angle\ D + \angle\ A,$ § 105

(*the exterior ∠ of a △ is equal to the sum of the two opposite interior* $\measuredangle$).

But $\angle\ D$ is measured by $\frac{1}{2}$ arc $A\ C$, § 203

(*an inscribed ∠ is measured by ½ the intercepted arc*) ;

and $\angle\ A$ is measured by $\frac{1}{2}$ arc $B\,D$, § 203

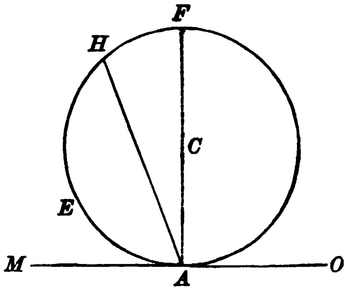
$\therefore \angle\ C\,O\,A$ is measured by $\frac{1}{2}$ arc $A\ C + \frac{1}{2}$ arc $B\,D$.

Q. E. D.

Ex. Show that the least chord that can be drawn through a given point in a circle is perpendicular to the diameter drawn through the point.

PROPOSITION XVI. THEOREM.

209. *An angle formed by a tangent and a chord is measured by one-half the intercepted arc.*



Let HAM be the angle formed by the tangent OM and chord AH.

We are to prove

$\angle HAM$ *is measured by* $\frac{1}{2}$ *arc* AEH.

Draw the diameter ACF.

$\angle FAM$ is a rt. $\angle$, § 186

(*the radius drawn to a tangent at the point of contact is* $\perp$ *to it*).

$\angle FAM$, being a rt. $\angle$, is measured by $\frac{1}{2}$ the semi-circumference AEF.

$\angle FAH$ is measured by $\frac{1}{2}$ arc FH, § 203

(*an inscribed* $\angle$ *is measured by* $\frac{1}{2}$ *the intercepted arc*);

$\therefore \angle FAM - \angle FAH$ is measured by $\frac{1}{2}$ (arc $AEF -$ arc HF).

$\therefore \angle HAM$ is measured by $\frac{1}{2}$ arc AEH.

Q. E. D.

PROPOSITION XVII. THEOREM.

210. ***An angle formed by two secants, two tangents, or a tangent and a secant, and which has its vertex without the circumference, is measured by one-half the concave arc, minus one-half the convex arc.***

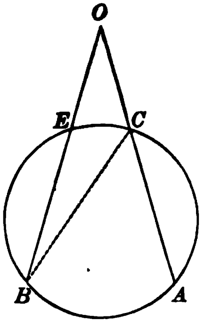


Fig. 1.

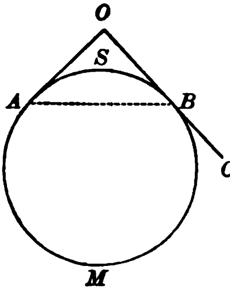


Fig. 2.

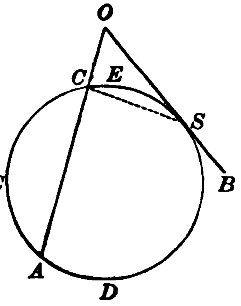


Fig. 3.

CASE I.

Let the angle O (Fig. 1) be formed by the two secants OA and OB.

We are to prove

$\angle O$ is measured by $\frac{1}{2}$ arc $AB - \frac{1}{2}$ arc EC.

Draw CB.

$\angle ACB = \angle O + \angle B$, § 105

(the exterior ∠ of a △ is equal to the sum of the two opposite interior ∠s).

By transposing,

$\angle O = \angle ACB - \angle B$,

But $\angle ACB$ is measured by $\frac{1}{2}$ arc AB, § 203

(an inscribed ∠ is measured by ½ the intercepted arc).

and $\angle B$ is measured by $\frac{1}{2}$ arc CE, § 203

$\therefore \angle O$ is measured by $\frac{1}{2}$ arc $AB - \frac{1}{2}$ arc CE.

CASE II.

Let the angle O (Fig. 2) be formed by the two tangents OA and OB.

We are to prove

$\angle O$ is measured by $\frac{1}{2}$ arc $AMB - \frac{1}{2}$ arc ASB.

Draw AB.

$\angle ABC = \angle O + \angle OAB$, § 105

(the exterior $\angle$ of a $\triangle$ is equal to the sum of the two opposite interior $\angle$s).

By transposing,

$$\angle O = \angle ABC - \angle OAB.$$

But $\angle ABC$ is measured by $\frac{1}{2}$ arc AMB, § 209

(an $\angle$ formed by a tangent and a chord is measured by $\frac{1}{2}$ the intercepted arc),

and $\angle OAB$ is measured by $\frac{1}{2}$ arc ASB. § 209

$\therefore \angle O$ is measured by $\frac{1}{2}$ arc $AMB - \frac{1}{2}$ arc ASB.

CASE III.

Let the angle O (Fig. 3) be formed by the tangent OB and the secant OA.

We are to prove

$\angle O$ is measured by $\frac{1}{2}$ arc $ADS - \frac{1}{2}$ arc CES.

Draw CS.

$\angle ACS = \angle O + \angle CSO$, § 105

(the exterior $\angle$ of a $\triangle$ is equal to the sum of the two opposite interior $\angle$s).

By transposing,

$$\angle O = \angle ACS - \angle CSO.$$

But $\angle ACS$ is measured by $\frac{1}{2}$ arc ADS, § 203

(being an inscribed $\angle$).

and $\angle CSO$ is measured by $\frac{1}{2}$ arc CES, § 209

(being an $\angle$ formed by a tangent and a chord).

$\therefore \angle O$ is measured by $\frac{1}{2}$ arc $ADS - \frac{1}{2}$ arc CES.

Q. E. D.

SUPPLEMENTARY PROPOSITIONS.

PROPOSITION XVIII. THEOREM.

211. *Two parallel lines intercept upon the circumference equal arcs.*

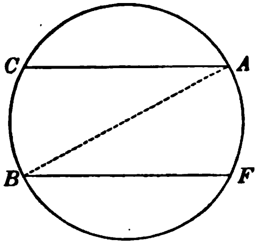


Fig. 1.

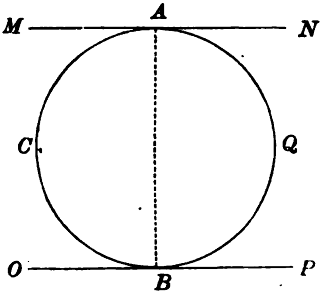


Fig. 2.

Let the two parallel lines CA and BF (Fig. 1), intercept the arcs CB and AF.

We are to prove arc CB = arc AF.

Draw AB.

$$\angle A = \angle B, \qquad \text{§ 68}$$

(*being alt.-int.* $\angle$s).

But the arc CB is double the measure of $\angle A$.

and the arc AF is double the measure of $\angle B$.

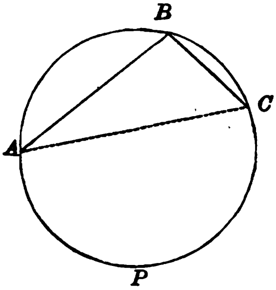
$\therefore$ arc CB = arc AF. Ax. 6.

Q. E. D.

212. SCHOLIUM. Since two parallel lines intercept on the circumference equal arcs, the two parallel tangents MN and OP (Fig. 2) divide the circumference in two semi-circumferences ACB and AQB, and the line AB joining the points of contact of the two tangents is a diameter of the circle.

PROPOSITION XIX. THEOREM.

213. *If the sum of two arcs be less than a circumference the greater arc is subtended by the greater chord; and conversely, the greater chord subtends the greater arc.*



In the circle ACP let the two arcs AB and BC together be less than the circumference, and let AB be the greater.

We are to prove chord $AB >$ chord BC.

Draw AC.

In the $\triangle ABC$

$\angle C$, measured by $\frac{1}{2}$ the greater arc AB, § 203

is greater than $\angle A$, measured by $\frac{1}{2}$ the less arc BC.

$\therefore$ the side $AB >$ the side BC, § 117

(in a $\triangle$ the greater $\angle$ has the greater side opposite to it).

CONVERSELY: If the chord AB be greater than the chord BC.

We are to prove arc $AB >$ arc BC.

In the $\triangle ABC$,

$AB > BC$, Hyp.

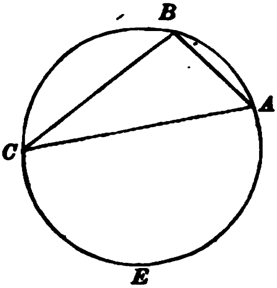
$\therefore \angle C > A$, § 118

(in a $\triangle$ the greater side has the greater $\angle$ opposite to it).

$\therefore$ arc AB, double the measure of the greater $\angle C$, is greater than the arc BC, double the measure of the less $\angle A$. Q. E. D.

PROPOSITION XX. THEOREM.

214. *If the sum of two arcs be greater than a circumference, the greater arc is subtended by the less chord; and, conversely, the less chord subtends the greater arc.*



In the circle BCE let the arcs $AECB$ and $BAEC$ together be greater than the circumference, and let arc $AECB$ be greater than arc $BAEC$.

We are to prove chord $AB <$ chord BC.

From the given arcs take the common arc AEC;

we have left two arcs, CB and AB, less than a circumference,

of which CB is the greater.

$\therefore$ chord $CB >$ chord AB, § 213

(*when the sum of two arcs is less than a circumference, the greater arc is subtended by the greater chord*).

$\therefore$ the chord AB, which subtends the greater arc $AECB$, is less than the chord BC, which subtends the less arc $BAEC$.

CONVERSELY: If the chord AB be less than chord BC.

We are to prove arc $AECB >$ arc $BAEC$.

Arc AB + arc $AECB$ = the circumference.

Arc BC + arc $BAEC$ = the circumference.

$\therefore$ arc AB + arc $AECB$ = arc BC + arc $BAEC$.

But arc $AB <$ arc BC, § 213

(*being subtended by the less chord*).

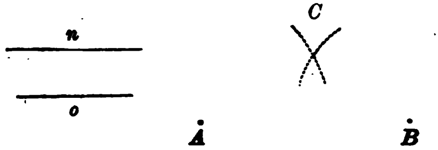
$\therefore$ arc $AECB >$ arc $BAEC$.

Q. E. D.

ON CONSTRUCTIONS.

PROPOSITION XXI. PROBLEM.

215. *To find a point in a plane, having given its distances from two known points.*



Let *A* and *B* be the two known points; *n* the distance of the required point from *A*, *o* its distance from *B*.

It is required to find a point at the given distances from A and B.

From *A* as a centre, with a radius equal to *n*, describe an arc.

From *B* as a centre, with a radius equal to *o*, describe an arc intersecting the former arc at *C*.

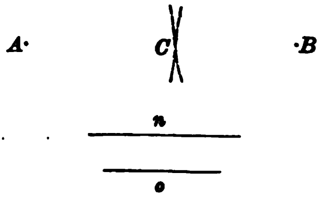
C is the required point.

Q. E. F.

216. COROLLARY 1. By continuing these arcs, another point below the points *A* and *B* will be found, which will fulfil the conditions.

217. COR. 2. When the sum of the given distances is equal to the distance between the two given points, then the two arcs described will be tangent to each other, and the point of tangency will be the point required.

Let the distance from A to B equal $n + o$.



From A as a centre, with a radius equal to n, describe an arc;

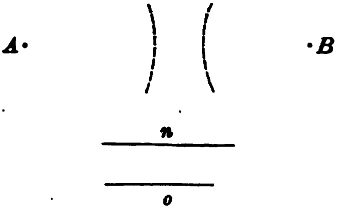
and from B as a centre, with a radius equal to o, describe an arc.

These arcs will touch each other at C, and will not intersect.

$\therefore$ C is the only point which can be found.

218. Scholium 1. The problem is impossible when the distance between the two known points is greater than the sum of the distances of the required point from the two given points.

Let the distance from A to B be greater than $n + o$.



Then from A as a centre, with a radius equal to n, describe an arc;

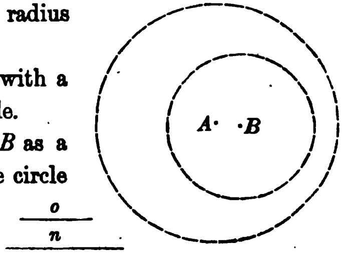
and from B as a centre, with a radius equal to o, describe an arc.

These arcs will neither touch nor intersect each other;

hence they can have no point in common.

219. Scho. 2. The problem is impossible when the distance between the two given points is less than the difference of the distances of the required point from the two given points.

Let the distance from A to B be less than $n - o$.



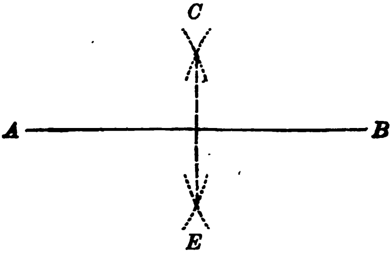
From A as a centre, with a radius equal to n, describe a circle;

and from B as a centre, with a radius equal to o, describe a circle.

The circle described from B as a centre will fall wholly within the circle described from A as a centre; hence they can have no point in common.

PROPOSITION XXII. PROBLEM.

220. *To bisect a given straight line.*



Let AB be the given straight line.

It is required to bisect the line AB.

From A and B as centres, with equal radii, describe arcs intersecting at C and E.

Join CE.

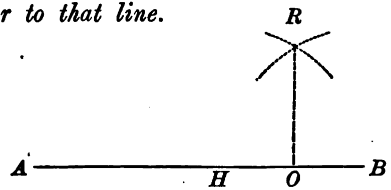
Then the line CE bisects AB.

For, C and E, being two points at equal distances from the extremities A and B, determine the position of a ⊥ to the middle point of AB. § 60

Q. E. F.

PROPOSITION XXIII. PROBLEM.

221. *At a given point in a straight line, to erect a perpendicular to that line.*



Let O be the given point in the straight line AB.

It is required to erect a ⊥ to the line AB at the point O.

Take $OH = OB$.

From B and H as centres, with equal radii, describe two arcs intersecting at R.

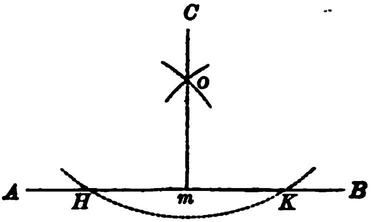
Then the line joining RO is the ⊥ required.

For, O and R are two points at equal distances from B and H, and ∴ determine the position of a ⊥ to the line HB at its middle point O. § 60

Q. E. F.

PROPOSITION XXIV. PROBLEM.

222. *From a point without a straight line, to let fall a perpendicular upon that line.*



Let $A B$ be a given straight line, and C a given point without the line.

It is required to let fall a ⊥ to the line $A B$ from the point C.

From C as a centre, with a radius sufficiently great,

describe an arc cutting $A B$ at the points H and K.

From H and K as centres, with equal radii,

describe two arcs intersecting at O.

Draw $C O$,

and produce it to meet $A B$ at m.

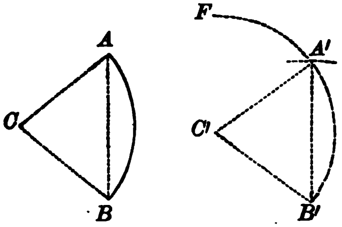
$C m$ is the ⊥ required.

For, C and O, being two points at equal distances from H and K, determine the position of a ⊥ to the line $H K$ at its middle point. § 60

Q. E. F.

PROPOSITION XXV. PROBLEM.

223. *To construct an arc equal to a given arc whose centre is a given point.*



Let C be the centre of the given arc AB.

It is required to construct an arc equal to arc AB.

Draw CB, CA, and AB.

From C' as a centre, with a radius equal to CB,

describe an indefinite arc $B'F$.

From B' as a centre, with a radius equal to chord AB,

describe an arc intersecting the indefinite arc at A'.

Then arc $A'B'$ = arc AB.

For, draw chord $A'B'$.

The ⓢ are equal,
(being described with equal radii),

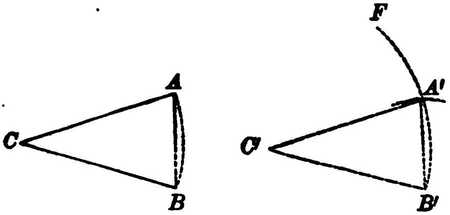
and chord $A'B'$ = chord AB; Cons.

∴ arc $A'B'$ = arc AB, § 182
(in equal ⓢ equal chords subtend equal arcs).

Q. E. F

PROPOSITION XXVI. PROBLEM.

224. ***At a given point in a given straight line to construct an angle equal to a given angle.***



Let C' be the given point in the given line $C'B'$, and C the given angle.

It is required to construct an $\angle$ at C' equal to the $\angle$ C.

From C as a centre, with any radius as CB,

describe the arc AB, terminating in the sides of the $\angle$.

Draw chord AB.

From C' as a centre, with a radius equal to CB,

describe the indefinite arc $B'F$.

From B' as a centre, with a radius equal to AB,

describe an arc intersecting the indefinite arc at A'.

Draw $A'C'$.

Then $\angle C' = \angle C$.

For, join $A'B'$.

The Ⓢ to which belong arcs AB and $A'B'$ are equal,
(being described with equal radii).

and chord $A'B'$ = chord AB; Cons.

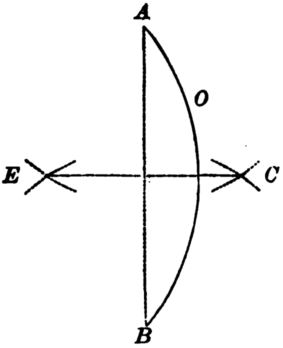
$\therefore$ arc $A'B'$ = arc AB, § 182
(in equal Ⓢ equal chords subtend equal arcs).

$\therefore \angle C' = \angle C$; § 180
(in equal Ⓢ equal arcs subtend equal ▵ at the centre).

Q. E. F.

PROPOSITION XXVII. PROBLEM.

225. *To bisect a given arc.*



Let $A\,O\,B$ be the given arc.

It is required to bisect the arc $A\,O\,B$.

Draw the chord $A\,B$.

From A and B as centres, with equal radii,

describe arcs intersecting at E and C.

Draw $E\,C$.

$E\,C$ bisects the arc $A\,O\,B$.

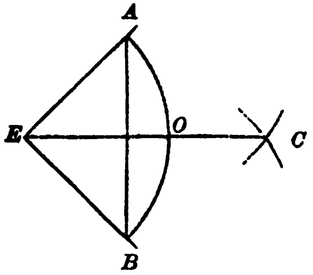
For, E and C, being two points at equal distances from A and B, determine the position of the ⊥ erected at the middle of chord $A\,B$; § 60

and a ⊥ erected at the middle of a chord passes through the centre of the ⊙, and bisects the arc of the chord. § 184

Q. E. F.

PROPOSITION XXVIII. PROBLEM.

226. *To bisect a given angle.*



Let $A E B$ be the given angle.

It is required to bisect $\angle A E B$.

From E as a centre, with any radius, as $E A$,

describe the arc $A O B$, terminating in the sides of the $\angle$.

Draw the chord $A B$.

From A and B as centres, with equal radii,

describe two arcs intersecting at C.

Join $E C$.

$E C$ bisects the $\angle E$.

For, E and C, being two points at equal distances from A and B, determine the position of the $\perp$ erected at the middle of $A B$. § 60

And the $\perp$ erected at the middle of a chord passes through the centre of the $\odot$, and bisects the arc of the chord. § 184

$\therefore$ arc $A O =$ arc $O B$,

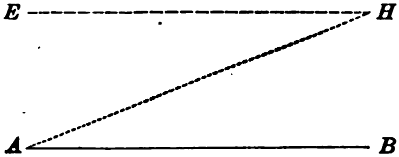
$\therefore \angle A E C = \angle B E C$, § 180

(in the same circle equal arcs subtend equal ∡ at the centre).

Q. E. F.

PROPOSITION XXIX. PROBLEM.

227. *Through a given point to draw a straight line parallel to a given straight line.*



Let AB be the given line, and H the given point.

It is required to draw through the point H a line ∥ to the line AB.

Draw HA, making the $\angle HAB$.

At the point H construct $\angle AHE = \angle HAB$.

Then the line HE is ∥ to AB.

For, $\angle EHA = \angle HAB$; Cons.

$\therefore HE$ is ∥ to AB, § 69

(when two straight lines, lying in the same plane, are cut by a third straight line, if the alt.-int. ∠s be equal, the lines are parallel).

Q. E. F.

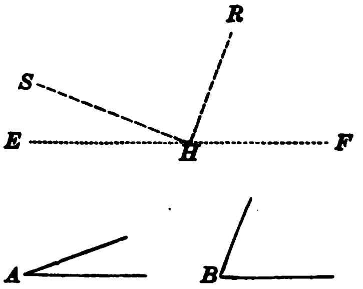
Ex. 1. Find the locus of the centre of a circumference which passes through two given points.

2. Find the locus of the centre of the circumference of a given radius, tangent externally or internally to a given circumference.

3. A straight line is drawn through a given point A, intersecting a given circumference at B and C. Find the locus of the middle point P of the intercepted chord BC.

PROPOSITION XXX. PROBLEM.

228. ***Two angles of a triangle being given to find the third.***



Let A and B be two given angles of a triangle.

It is required to find the third ∠ of the △.

Take any straight line, as *E F*, and at any point, as *H*.

construct ∠ *R H F* equal to ∠ *B*,

and ∠ *S H E* equal to ∠ *A*.

Then ∠ *R H S* is the ∠ required.

For, the sum of the three ∠s of a △ = 2 rt. ∠s, § 98

and the sum of the three ∠s about the point *H*, on the same side of *E F* = 2 rt. ∠s. § 34

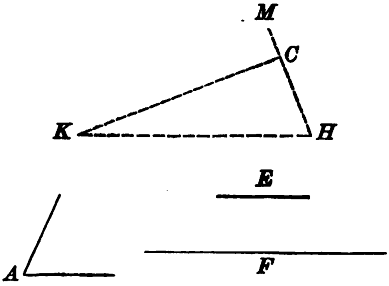
Two ∠s of the △ being equal to two ∠s about the point *H*, Cons.

the third ∠ of the △ must be equal to the third ∠ about the point *H*.

Q. E. F.

Proposition XXXI. Problem.

229. *Two sides and the included angle of a triangle being given, to construct the triangle.*



Let the two sides of the triangle be E and F, and the included angle A.

It is required to construct a △ having two sides equal to E and F respectively, and their included ∠ = ∠ A.

Take HK equal to the side F.

At the point H draw the indefinite line HM,

making the $\angle KHM = \angle A$.

On HM take HC equal to E.

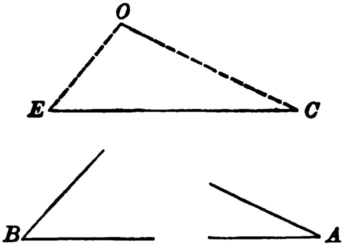
Draw CK.

Then $\triangle CHK$ is the △ required.

Q. E. F.

PROPOSITION XXXII. PROBLEM.

230. ***A side and two adjacent angles of a triangle being given, to construct the triangle.***



Let *C E* be the given side, *A* and *B* the given angles.

It is required to construct a △ having a side equal to C E, and two △s adjacent to that side equal to △s A and B respectively.

At point *C* construct an ∠ equal to ∠ *A*.

At point *E* construct an ∠ equal to ∠ *B*.

Produce the sides until they meet at *O*.

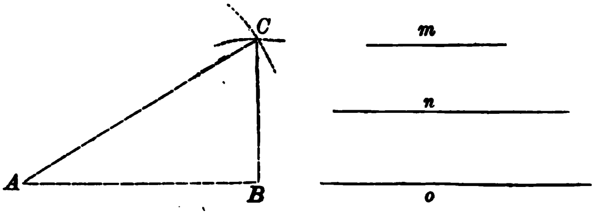
Then △ *C O E* is the △ required.

Q. E. F.

231. SCHOLIUM. The problem is impossible when the two given angles are together equal to, or greater than, two right angles.

PROPOSITION XXXIII. PROBLEM.

232. *The three sides of a triangle being given, to construct the triangle.*



Let the three sides be *m*, *n*, and *o*.

It is required to construct a △ having three sides respectively, equal to m, n, and o.

Draw AB equal to n.

From A as a centre, with a radius equal to o,

describe an arc;

and from B as a centre, with a radius equal to m,

describe an arc intersecting the former arc at C.

Draw CA and CB.

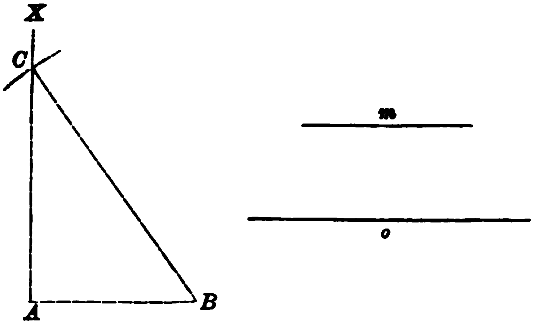
Then $\triangle CAB$ is the $\triangle$ required.

Q. E. F.

233. SCHOLIUM. The problem is impossible when one side is *equal to or greater than the sum of the other two.*

PROPOSITION XXXIV. PROBLEM.

234. *The hypotenuse and one side of a right triangle being given, to construct the triangle.*



Let *m* be the given side, and *o* the hypotenuse.

It is required to construct a rt. △ *having the hypotenuse equal o and one side equal m.*

Take $A\,B$ equal to m.

At A erect a ⊥, $A\,X$.

From B as a centre, with a radius equal to o,

describe an arc cutting $A\,X$ at C.

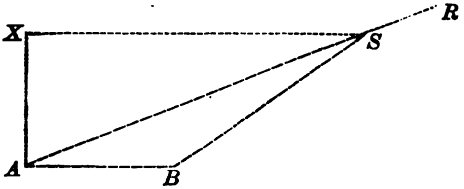
Draw $C\,B$.

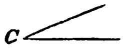
Then △ $C\,A\,B$ is the △ required.

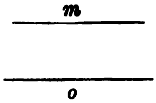
Q. E. F.

PROPOSITION XXXV. PROBLEM.

235. *The base, the altitude, and an angle at the base, of a triangle being given, to construct the triangle.*







Let o equal the base, m the altitude, and C the angle at the base.

It is required to construct a △ having the base equal to o, the altitude equal to m, and an ∠ at the base equal to C.

Take AB equal to o.

At the point A, draw the indefinite line AR,

making the $\angle BAR = \angle C$.

At the point A, erect a $\perp AX$ equal to m.

From X draw $XS \parallel$ to AB,

and meeting the line AR at S.

Draw SB.

Then △ ASB is the △ required.

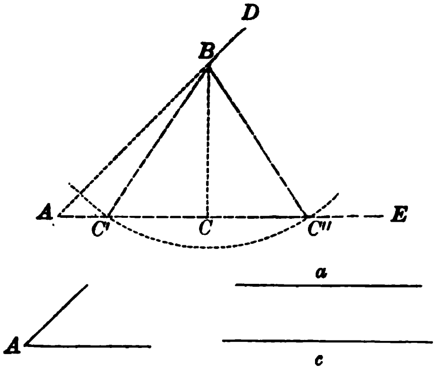
Q. E. F.

PROPOSITION XXXVI. PROBLEM.

236. ***Two sides of a triangle and the angle opposite one of them being given, to construct the triangle.***

CASE I.

When the given angle is acute, and the side opposite to it is less than the other given side.



Let c **be the longer and** a **the shorter given side, and** $\angle A$ **the given angle.**

It is required to construct a $\triangle$ *having two sides equal to* a *and* c *respectively, and the* $\angle$ *opposite* a *equal to given* $\angle A$.

Construct $\angle DAE$ equal to the given $\angle A$.

On AD take $AB = c$.

From B as a centre, with a radius equal to a,

describe an arc intersecting the side AE at C' and C''.

Draw BC' and BC''.

Then both the $\triangle$s ABC' and ABC'' fulfil the conditions, and hence we have two constructions.

When the given side a is exactly equal to the $\perp$ BC, there will be but one construction, namely, the right triangle ABC.

When the given side a is less than BC, the arc described from B will not intersect AE, and hence the problem is impossible.

CASE II.

When the given angle is acute, right, or obtuse, and the side opposite to it is greater than the other given side.

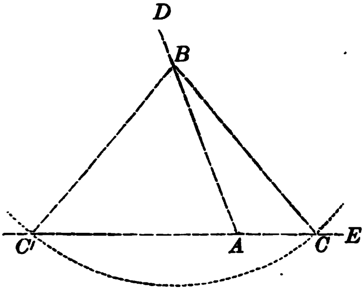


Fig. 1.

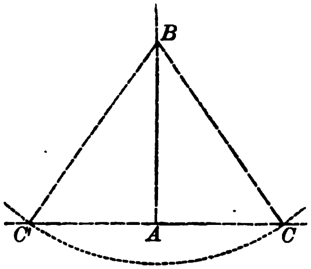
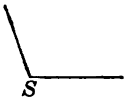
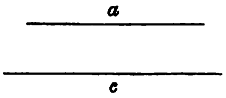


Fig. 2.





When the given angle is obtuse.

Construct the $\angle DAE$ (Fig. 1) equal to the given $\angle S$.

Take AB equal to a.

From B as a centre, with a radius equal to c,

describe an arc cutting EA at C, and EA produced at C'.

Join BC and BC'.

Then the $\triangle ABC$ is the $\triangle$ required, and there is only one construction; for the $\triangle ABC'$ will not contain the given $\angle S$.

When the given angle is acute, as angle BAC'.

There is only one construction, namely, the $\triangle BAC'$ (Fig. 1).

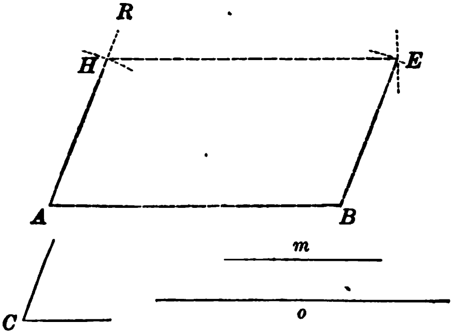
When the given $\angle$ is a right angle.

There are two constructions, the equal △s BAC and BAC' (Fig. 2). Q. E. F.

The problem is impossible when the given angle is right or obtuse, if the given side opposite the angle be less than the other given side. § 117

Proposition XXXVII. Problem.

237. *Two sides and an included angle of a parallelogram being given, to construct the parallelogram.*



Let m and o be the two sides, and C the included angle.

It is required to construct a ▱ having two adjacent sides equal to m and o respectively, and their included ∠ equal to ∠ C.

Draw AB equal to o.

From A draw the indefinite line AR, making the ∠ A equal to ∠ C.

On AR take AH equal to m.

From H as a centre, with a radius equal to o, describe an arc.

From B as a centre, with a radius equal to m, describe an arc, intersecting the former arc at E.

Draw EH and EB.

The quadrilateral $ABEH$ is the ▱ required.

For, $AB = HE$, Cons.

$AH = BE$, Cons.

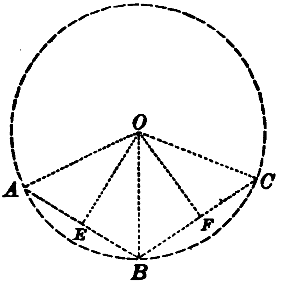
∴ the figure $ABEH$ is a ▱, § 136

(a quadrilateral, which has its opposite sides equal, is a ▱).

Q. E. F.

PROPOSITION XXXVIII. PROBLEM.

238. *To describe a circumference through three points not in the same straight line.*



Let the three points be A, B, and C.

It is required to describe a circumference through the three points A, B, and C.

Draw AB and BC.

Bisect AB and BC.

At the points of bisection, E and F, erect $\perp$s intersecting at O.

From O as a centre, with a radius equal to OA, describe a circle.

$\odot\, ABC$ is the $\odot$ required.

For, the point O, being in the $\perp$ EO erected at the middle of the line AB, is at equal distances from A and B;

and also, being in the $\perp$ FO erected at the middle of the line CB, is at equal distances from B and C, § 58

(every point in the $\perp$ erected at the middle of a straight line is at equal distances from the extremities of that line).

$\therefore$ the point O is at equal distances from A, B, and C,

and a $\odot$ described from O as a centre, with a radius equal to OA, will pass through the points A, B, and C.

Q. E. F.

239. SCHOLIUM. The same construction serves to describe a circumference which shall pass through the three vertices of a triangle, that is, to circumscribe a circle about a given triangle.

PROPOSITION XXXIX. PROBLEM.

240. *Through a given point to draw a tangent to a given circle.*

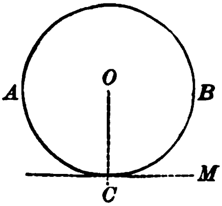


Fig. 1.

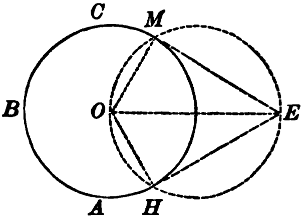


Fig. 2.

CASE 1.— *When the given point is on the circumference.*

Let ABC (Fig. 1) be a given circle, and C the given point on the circumference.

It is required to draw a tangent to the circle at C.

From the centre O, draw the radius OC.

At the extremity of the radius, C, draw $CM \perp$ to OC.

Then CM is the tangent required, § 186

(a straight line $\perp$ to a radius at its extremity is tangent to the $\odot$).

CASE 2.— *When the given point is without the circumference.*

Let ABC (Fig. 2) be the given circle, O its centre, E the given point without the circumference.

It is required to draw a tangent to the circle ABC from the point E.

Join OE.

On OE as a diameter, describe a circumference intersecting the given circumference at the points M and H.

Draw OM and OH, EM and EH.

Now $\angle OME$ is a rt. $\angle$, . § 204

(being inscribed in a semicircle).

$\therefore EM$ is $\perp$ to OM at the point M;

$\therefore EM$ is tangent to the $\odot$, § 186

(a straight line $\perp$ to a radius at its extremity is tangent to the $\odot$).

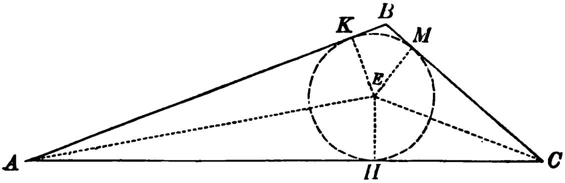
In like manner we may prove HE tangent to the given $\odot$.

Q. E. F.

241. COROLLARY. Two tangents drawn from the same point to a circle are equal.

PROPOSITION XL. PROBLEM.

242. ***To inscribe a circle in a given triangle.***



Let ABC be the given triangle.

It is required to inscribe a ⊙ in the △ ABC.

Draw the line AE, bisecting $\angle A$,

and draw the line CE, bisecting $\angle C$.

Draw $EH \perp$ to the line AC.

From E, with radius EH, describe the ⊙ KMH.

The ⊙ KHM is the ⊙ required.

For, draw $EK \perp$ to AB,

and $EM \perp$ to BC.

In the rt. ⊿ AKE and AHE

$AE = AE$, Iden.

$\angle EAK = \angle EAH$, Cons.

$\therefore \triangle AKE = \triangle AHE$, § 110

(*Two rt. ⊿ are equal if the hypotenuse and an acute ∠ of the one be equal respectively to the hypotenuse and an acute ∠ of the other*).

$\therefore EK = EH$,

(*being homologous sides of equal ⊿*).

In like manner it may be shown $EM = EH$.

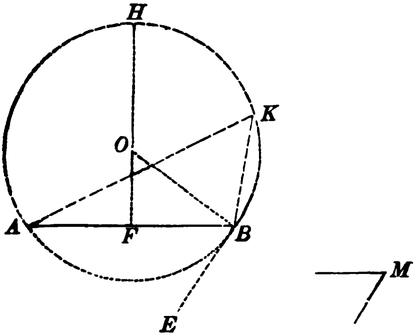
$\therefore EK$, EH, and EM are all equal.

$\therefore$ a ⊙ described from E as a centre, with a radius equal to EH, will touch the sides of the △ at points H, K, and M, and be inscribed in the △. § 174

Q. E. F.

PROPOSITION XLI. PROBLEM.

243. ***Upon a given straight line, to describe a segment which shall contain a given angle.***



Let $A B$ be the given line, and M the given angle.

It is required to describe a segment upon the line $A B$, which shall contain $\angle M$.

At the point B construct $\angle A B E$ equal to $\angle M$.

Bisect the line $A B$ at F,

and erect the $\perp$ $F H$.

From the point B, draw $B O \perp$ to $E B$.

From O, the point of intersection of $F H$ and $B O$, as a centre, with a radius equal to $O B$, describe a circumference.

Now the point O, being in a $\perp$ erected at the middle of $A B$, is at equal distances from A and B, § 58

(*every point in a $\perp$ erected at the middle of a straight line is at equal distances from the extremities of that line*);

$\therefore$ the circumference will pass through A.

Now $B E$ is $\perp$ to $O B$, Cons.

$\therefore B E$ is tangent to the $\odot$, § 186

(*a straight line $\perp$ to a radius at its extremity is tangent to the $\odot$*).

$\therefore \angle A B E$ is measured by $\frac{1}{2}$ arc $A B$, § 209

(*being an $\angle$ formed by a tangent and a chord*).

Also any $\angle$ inscribed in the segment $A H B$, as for instance $\angle A K B$, is measured by $\frac{1}{2}$ arc $A B$, § 203

(*being an inscribed $\angle$*).

$$\therefore \angle AKB = \angle ABE,$$

(being both measured by ½ the same arc);

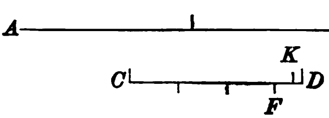
$$\therefore \angle AKB = \angle M.$$

∴ segment AHB is the segment required.

Q. E. F.

Proposition XLII. Problem.

244. *To find the ratio of two commensurable straight lines.*



Let AB and CD be two straight lines.

It is required to find the greatest common measure of AB and CD, so as to express their ratio in figures.

Apply CD to AB as many times as possible.

Suppose twice with a remainder EB.

Then apply EB to CD as many times as possible.

Suppose three times with a remainder FD.

Then apply FD to EB as many times as possible.

Suppose once with a remainder HB.

Then apply HB to FD as many times as possible.

Suppose once with a remainder KD.

Then apply KD to HB as many times as possible.

Suppose KD is contained just twice in HB.

The measure of each line, referred to KD as a unit, will then be as follows: —

$$\begin{aligned} HB &= 2KD; \\ FD &= HB + KD = 3KD; \\ EB &= FD + HB = 5KD; \\ CD &= 3EB + FD = 18KD; \\ AB &= 2CD + EB = 41KD. \end{aligned}$$

$$\therefore \frac{AB}{CD} = \frac{41KD}{18KD};$$

$$\therefore \text{the ratio of } \frac{AB}{CD} = \frac{41}{18}.$$

Q. E. F.

EXERCISES.

1. If the sides of a pentagon, no two sides of which are parallel, be produced till they meet; show that the sum of all the angles at their points of intersection will be equal to two right angles.

2. Show that two chords which are equally distant from the centre of a circle are equal to each other; and of two chords, that which is nearer the centre is greater than the one more remote.

3. If through the angles of an isosceles triangle which has each of the angles at the base double of the third angle, and is inscribed in a circle, straight lines be drawn touching the circle; show that an isosceles triangle will be formed which has each of the angles at the base one-third of the angle at the vertex.

4. ADB is a semicircle of which the centre is C; and AEC is another semicircle on the diameter AC; AT is a common tangent to the two semicircles at the point A. Show that if from any point F, in the circumference of the first, a straight line FC be drawn to C, the part FK, cut off by the second semicircle, is equal to the perpendicular FH to the tangent AT.

5. Show that the bisectors of the angles contained by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.

6. If a triangle ABC be formed by the intersection of three tangents to a circumference whose centre is O, two of which, AM and AN, are fixed, while the third, BC, touches the circumference at a variable point P; show that the perimeter of the triangle ABC is constant, and equal to $AM + AN$, or $2AM$. Also show that the angle BOC is constant.

7. AB is any chord and AC is tangent to a circle at A, CDE a line cutting the circumference in D and E and parallel to AB; show that the triangle ACD is equiangular to the triangle EAB.

CONSTRUCTIONS.

1. Draw two concentric circles, such that the chords of the outer circle which touch the inner may be equal to the diameter of the inner circle.

2. Given the base of a triangle, the vertical angle, and the length of the line drawn from the vertex to the middle point of the base: construct the triangle.

3. Given a side of a triangle, its vertical angle, and the radius of the circumscribing circle: construct the triangle.

4. Given the base, vertical angle, and the perpendicular from the extremity of the base to the opposite side: construct the triangle.

5. Describe a circle cutting the sides of a given square, so that its circumference may be divided at the points of intersection into eight equal arcs.

6. Construct an angle of 60°, one of 30°, one of 120°, one of 150°, one of 45°, and one of 135°.

7. In a given triangle $A B C$, draw $Q D E$ parallel to the base $B C$ and meeting the sides of the triangle at D and E, so that $D E$ shall be equal to $D B + E C$.

8. Given two perpendiculars, $A B$ and $C D$, intersecting in O, and a straight line intersecting these perpendiculars in E and F; to construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie in $E F$. (Two solutions.)

9. In a given rhombus to inscribe a square.

10. If the base and vertical angle of a triangle be given; find the locus of the vertex.

11. If a ladder, whose foot rests on a horizontal plane and top against a vertical wall, slip down; find the locus of its middle point.

BOOK III.

PROPORTIONAL LINES AND SIMILAR POLYGONS.

ON THE THEORY OF PROPORTION.

245. **DEF.** The *Terms* of a ratio are the quantities compared.

246. **DEF.** The *Antecedent of a ratio* is its first term.

247. **DEF.** The *Consequent of a ratio* is its second term.

248. **DEF.** A *Proportion* is an expression of equality between two equal ratios.

A proportion may be expressed in any one of the following forms:—

1. $a : b :: c : d$
2. $a : b = c : d$
3. $\frac{a}{b} = \frac{c}{d}.$

Form 1 is read, a is to b as c is to d.

Form 2 is read, the ratio of a to b equals the ratio of c to d.

Form 3 is read, a divided by b equals c divided by d.

The *Terms* of a proportion are the four quantities compared.

The *first* and *third* terms in a proportion are the antecedents, the *second* and *fourth* terms are the consequents.

249. The *Extremes* in a proportion are the *first* and *fourth* terms.

250. The *Means* in a proportion are the *second* and *third* terms.

251. **Def.** In the proportion $a : b :: c : d$; d is a *Fourth Proportional* to a, b, and c.

252. **Def.** In the proportion $a : b :: b : c$; c is a *Third Proportional* to a and b.

253. **Def.** In the proportion $a : b :: b : c$; b is a *Mean Proportional* between a and c.

254. **Def.** Four quantities are *Reciprocally Proportional* when the first is to the second as the reciprocal of the third is to the reciprocal of the fourth.

Thus $$a : b :: \frac{1}{c} : \frac{1}{d}.$$

If we have two quantities a and b, and the reciprocals of these quantities $\frac{1}{a}$ and $\frac{1}{b}$; these four quantities form a *reciprocal proportion*, the first being to the second as the reciprocal of the second is to the reciprocal of the first.

As $$a : b :: \frac{1}{b} : \frac{1}{a}.$$

255. **Def.** A proportion is taken by *Alternation*, when the means, or the extremes, are made to exchange places.

Thus in the proportion

$$a : b :: c : d,$$

we have either

$$a : c :: b : d, \text{ or, } d : b :: c : a.$$

256. **Def.** A proportion is taken by *Inversion*, when the means and extremes are made to exchange places.

Thus in the proportion

$$a : b :: c : d,$$

by inversion we have

$$b : a :: d : c.$$

257. **Def.** A proportion is taken by *Composition*, when the sum of the first and second is to the second as the sum of

the third and fourth is to the fourth; or when the sum of the first and second is to the first as the sum of the third and fourth is to the third.

Thus if $a : b :: c : d$,

we have by composition,

$$a + b : b :: c + d : d,$$

or, $$a + b : a :: c + d : c.$$

258. Def. A proportion is taken by *Division*, when the difference of the first and second is to the second as the difference of the third and fourth is to the fourth; or when the difference of the first and second is to the first as the difference of the third and fourth is to the third.

Thus if $a : b :: c : d$,

we have by division

$$a - b : b :: c - d : d,$$

or, $$a - b : a :: c - d : c.$$

Proposition I.

259. *In every numerical proportion the product of the extremes is equal to the product of the means.*

Let $a : b :: c : d$.

We are to prove $ad = bc$.

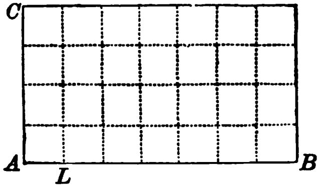
Now $$\frac{a}{b} = \frac{c}{d},$$

whence, by multiplying by bd,

$$ad = bc.$$

Q. E. D.

It is to be observed that the product of two quantities implies that at least one of the quantities is an abstract number. We cannot multiply dollars by dollars, or lines by lines. Still we may speak of the product of two lines, if we understand by the expression the product of the numbers which represent the lines when they are measured by a common unit.



Thus in the rectangle of the lines *A B* and *A C*, suppose *A L* to be a common measure of *A B* and *A C*, and to be contained in *A B* seven times, and in *A C* four times.

At the several points of division on *A C* draw lines perpendicular to *A C*,

and at the several points of division on *A B* draw lines perpendicular to *A B*.

Then the rectangle will be divided into squares, all equal to each other. There will be seven in a row, and four rows; that is *four times seven squares.*

Proposition II.

260. *A mean proportional between two quantities is equal to the square root of their product.*

In the proportion $a : b :: b : c$,

$$b^2 = ac, \qquad \text{§ 259}$$

(*the product of the extremes is equal to the product of the means*).

Whence, extracting the square root,

$$b = \sqrt{ac.}$$

Q. E. D.

PROPOSITION III.

261. *If the product of two quantities be equal to the product of two others, either two may be made the extremes of a proportion in which the other two are made the means.*

Let $ad = bc$.

We are to prove $a : b :: c : d$.

Divide both members of the given equation by bd.

Then $$\frac{a}{b} = \frac{c}{d},$$

or, $$a : b :: c : d.$$

Q. E. D.

PROPOSITION IV.

262. *If four quantities of the same kind be in proportion, they will be in proportion by alternation.*

Let $a : b :: c : d$.

We are to prove $a : c :: b : d$.

Now, $$\frac{a}{b} = \frac{c}{d}.$$

Multiply each member of the equation by $\frac{b}{c}$.

Then $$\frac{a}{c} = \frac{b}{d},$$

or, $$a : c :: b : d.$$

Q. E. D.

PROPOSITION V.

263. *If four quantities be in proportion, they will be in proportion by inversion.*

Let $a : b :: c : d.$

We are to prove $b : a :: d : c.$

Now, $$\frac{a}{b} = \frac{c}{d}.$$

Divide 1 by each member of the equation.

Then $$\frac{b}{a} = \frac{d}{c},$$

or, $$b : a :: d : c.$$

Q. E. D.

PROPOSITION VI.

264. *If four quantities be in proportion, they will be in proportion by composition.*

Let $a : b :: c : d$

We are to prove $a + b : b :: c + d : d.$

Now $$\frac{a}{b} = \frac{c}{d}.$$

Add 1 to each member of the equation.

Then $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

that is, $$\frac{a+b}{b} = \frac{c+d}{d},$$

or, $$a + b : b :: c + d : d.$$

Q. E. D.

PROPOSITION VII.

265. *If four quantities be in proportion, they will be in proportion by division.*

Let $a : b :: c : d.$

We are to prove $a - b : b :: c - d : d.$

Now $\frac{a}{b} = \frac{c}{d}.$

Subtract 1 from each member of the equation.

Then $\frac{a}{b} - 1 = \frac{c}{d} - 1,$

that is, $\frac{a-b}{b} = \frac{c-d}{d},$

or, $a - b : b :: c - d : d.$

Q. E. D.

PROPOSITION VIII.

266. *In a series of equal ratios, of which all the terms are of the same kind, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let $a : b = c : d = e : f = g : h.$

We are to prove $a + c + e + g : b + d + f + h :: a : b.$

Denote each ratio by r.

Then $r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}.$

Whence, $a = br,$ $c = dr,$ $e = fr,$ $g = hr.$

Add these equations.

Then $a + c + e + g = (b + d + f + h)\,r.$

Divide by $(b + d + f + h).$

Then $\frac{a+c+e+g}{b+d+f+h} = r = \frac{a}{b},$

or, $a + c + e + g : b + d + f + h :: a : b.$

Q. E. D.

PROPOSITION IX.

267. *The products of the corresponding terms of two or more numerical proportions are in proportion.*

$$\textit{Let } a : b :: c : d,$$
$$e : f :: g : h,$$
$$k : l :: m : n,$$

We are to prove $aek : bfl :: cgm : dhn.$

Now $\frac{a}{b} = \frac{c}{d}, \quad \frac{e}{f} = \frac{g}{h}, \quad \frac{k}{l} = \frac{m}{n}.$

Whence by multiplication,

$$\frac{aek}{bfl} = \frac{cgm}{dhn},$$

or, $aek : bfl :: cgm : dhn.$

Q. E. D.

PROPOSITION X.

268. *Like powers, or like roots, of a numerical proportion are in proportion.*

$$\textit{Let } a : b :: c : d.$$

We are to prove $a^n : b^n :: c^n : d^n,$

and $a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$

Now $\frac{a}{b} = \frac{c}{d}.$

By raising to the n^{th} power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n}; \text{ or } a^n : b^n :: c^n : d^n.$$

By extracting the n^{th} root,

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}; \text{ or, } a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

Q. E. D.

269. **DEF.** *Equimultiples* of two quantities are the products obtained by multiplying each of them by the same number. Thus ma and mb are equimultiples of a and b.

PROPOSITION XI.

270. *Equimultiples of two quantities are in the same ratio as the quantities themselves.*

Let a and b be any two quantities.

We are to prove $ma : mb :: a : b$.

Now
$$\frac{a}{b} = \frac{a}{b}.$$

Multiply both terms of first fraction by m.

Then
$$\frac{ma}{mb} = \frac{a}{b},$$

or,
$$ma : mb :: a : b.$$

Q. E. D.

PROPOSITION XII.

271. *If two quantities be increased or diminished by like parts of each, the results will be in the same ratio as the quantities themselves.*

Let a and b be any two quantities.

We are to prove $a \pm \frac{p}{q} a : b \pm \frac{p}{q} b :: a : b$.

In the proportion,
$$ma : mb :: a : b,$$

substitute for m, $1 \pm \frac{p}{q}$.

Then
$$\left(1 \pm \frac{p}{q}\right) a : \left(1 \pm \frac{p}{q}\right) b :: a : b,$$

or
$$a \pm \frac{p}{q} a : b \pm \frac{p}{q} b :: a : b.$$

Q. E. D.

272. DEF. Euclid's test of a proportion is as follows: —

"The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth;

"If the multiple of the first be *less* than that of the second, the multiple of the third is also less than that of the fourth; or,

"If the multiple of the first be *equal* to that of the second, the multiple of the third is also equal to that of the fourth; or,

"If the multiple of the first be *greater* than that of the second, the multiple of the third is also greater than that of the fourth."

PROPOSITION XIII.

273. ***If four quantities be proportional according to the algebraical definition, they will also be proportional according to the geometrical definition.***

***Let** a, b, c, d **be proportional according to the algebraical definition; that is** $\frac{a}{b} = \frac{c}{d}$.*

We are to prove a, b, c, d, *proportional according to the geometrical definition.*

Multiply each member of the equality by $\frac{m}{n}$.

Then $$\frac{ma}{nb} = \frac{mc}{nd}.$$

Now from the nature of fractions,

if ma be less than nb, mc will also be less than nd;

if ma be equal to nb, mc will also be equal to nd;

if ma be greater than nb, mc will also be greater than nd.

∴ a, b, c, d are proportionals according to the geometrical definition.

Q. E. D.

EXERCISES.

1. Show that the straight line which bisects the external vertical angle of an isosceles triangle is parallel to the base.

2. A straight line is drawn terminated by two parallel straight lines; through its middle point any straight line is drawn and terminated by the parallel straight lines. Show that the second straight line is bisected at the middle point of the first.

3. Show that the angle between the bisector of the angle *A* of the triangle *A B C* and the perpendicular let fall from *A* on *B C* is equal to one-half the difference between the angles *B* and *C*.

4. In any right triangle show that the straight line drawn from the vertex of the right angle to the middle of the hypotenuse is equal to one-half the hypotenuse.

5. Two tangents are drawn to a circle at opposite extremities of a diameter, and cut off from a third tangent a portion *A B*. If *C* be the centre of the circle, show that *A C B* is a right angle.

6. Show that the sum of the three perpendiculars from any point within an equilateral triangle to the sides is equal to the altitude of the triangle.

7. Show that the least chord which can be drawn through a given point within a circle is perpendicular to the diameter drawn through the point.

8. Show that the angle contained by two tangents at the extremities of a chord is twice the angle contained by the chord and the diameter drawn from either extremity of the chord.

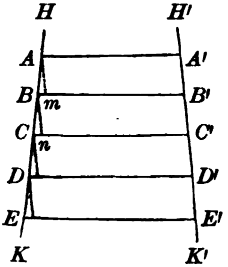
9. If a circle can be inscribed in a quadrilateral; show that the sum of two opposite sides of the quadrilateral is equal to the sum of the other two sides.

10. If the sum of two opposite sides of a quadrilateral be equal to the sum of the other two sides; show that a circle can be inscribed in the quadrilateral.

ON PROPORTIONAL LINES.

PROPOSITION I. THEOREM.

274. *If a series of parallels intersecting any two straight lines intercept equal parts on one of these lines, they will intercept equal parts on the other also.*



Let the series of parallels AA', BB', CC', DD', EE', intercept on $H'K'$ equal parts $A'B'$, $B'C'$, $C'D'$, etc.

We are to prove
they intercept on HK equal parts AB, BC, CD, etc.

At points A and B draw Am and Bn ∥ to $H'K'$.

$$Am = A'B', \qquad \S\ 135$$

(*parallels comprehended between parallels are equal*).

$$Bn = B'C', \qquad \S\ 135$$

$$\therefore Am = Bn.$$

In the △ BAm and CBn,

$$\angle A = \angle B, \qquad \S\ 77$$

(*having their sides respectively* ∥ *and lying in the same direction from the vertices*).

$$\angle m = \angle n, \qquad \S\ 77$$

and $$Am = Bn,$$

$$\therefore \triangle BAm = \triangle CBn, \qquad \S\ 107$$

(*having a side and two adj.* ∠ *of the one equal respectively to a side and two adj.* ∠ *of the other*).

$$\therefore AB = BC,$$

(*being homologous sides of equal* △).

In like manner we may prove $BC = CD$, etc.

Q. E. D.

PROPOSITION II. THEOREM.

275. *If a line be drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally.*

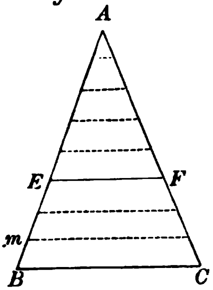


Fig. 1.

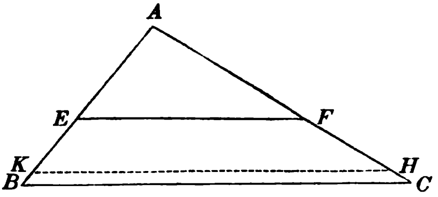


Fig. 2.

In the triangle ABC let EF be drawn parallel to BC.

We are to prove $\dfrac{EB}{AE} = \dfrac{FC}{AF}$.

CASE I. — *When AE and EB (Fig. 1) are commensurable.*

Find a common measure of AE and EB, namely Bm.

Suppose Bm to be contained in BE three times,

and in AE five times.

Then $$\frac{EB}{AE} = \frac{3}{5}.$$

At the several points of division on BE and AE draw straight lines ∥ to BC.

These lines will divide AC into eight equal parts,

of which FC will contain three, and AF will contain five, § 274

(*if parallels intersecting any two straight lines intercept equal parts on one of these lines, they will intercept equal parts on the other also*).

$$\therefore \frac{FC}{AF} = \frac{3}{5}.$$

But $$\frac{EB}{AE} = \frac{3}{5},$$

$$\therefore \frac{EB}{AE} = \frac{FC}{AF}. \qquad \text{Ax. 1}$$

CASE. II. — *When AE and EB (Fig. 2) are incommensurable.*

Divide AE into any number of equal parts,

and apply one of these parts to EB as often as it will be contained in EB.

Since AE and EB are incommensurable, a certain number of these parts will extend from E to a point K, leaving a remainder KB, less than one of the parts.

Draw $KH \parallel$ to BC.

Since AE and EK are commensurable,

$$\frac{EK}{AE} = \frac{FH}{AF} \qquad \text{(Case I.)}$$

Suppose the number of parts into which AE is divided to be continually increased, the length of each part will become less and less, and the point K will approach nearer and nearer to B.

The limit of EK will be EB, and the limit of FH will be FC.

$$\therefore \text{ the limit of } \frac{EK}{AE} \text{ will be } \frac{EB}{AE},$$

and

$$\text{the limit of } \frac{FH}{AF} \text{ will be } \frac{FC}{AF}.$$

Now the variables $\frac{EK}{AE}$ and $\frac{FH}{AF}$ are always equal, however near they approach their limits;

$$\therefore \text{ their limits } \frac{EB}{AE} \text{ and } \frac{FC}{AF} \text{ are equal,} \qquad \S\ 199$$

Q. E. D.

276. COROLLARY. One side of a triangle is to either part cut off by a straight line parallel to the base, as the other side is to the corresponding part.

Now $EB : AE :: FC : AF$. § 275

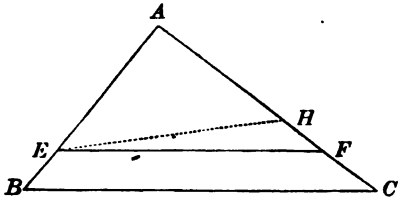
By composition,

$EB + AE : AE :: FC + AF : AF$, § 263

or, $AB : AE :: AC : AF$.

PROPOSITION III. THEOREM.

277. *If a straight line divide two sides of a triangle proportionally, it is parallel to the third side.*



In the triangle ABC let EF be drawn so that $\frac{AB}{AE} = \frac{AC}{AF}$.

We are to prove $EF \parallel$ *to* BC.

From E draw $EH \parallel$ to BC.

Then $$\frac{AB}{AE} = \frac{AC}{AH}, \qquad \text{§ 276}$$

(one side of a △ is to either part cut off by a line ∥ to the base, as the other side is to the corresponding part).

But $$\frac{AB}{AE} = \frac{AC}{AF}, \qquad \text{Hyp.}$$

$$\therefore \frac{AC}{AF} = \frac{AC}{AH}, \qquad \text{Ax. 1}$$

$$\therefore AF = AH.$$

$\therefore$ EF and EH coincide,

(their extremities being the same points).

But EH is $\parallel$ to BC; Cons.

$\therefore$ EF, which coincides with EH, is $\parallel$ to BC.

Q. E. D.

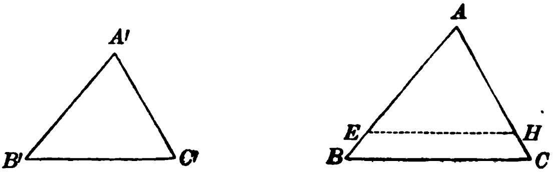
278. DEF. *Similar Polygons* are polygons which have their homologous angles equal and their homologous sides proportional.

Homologous points, lines, and angles, in similar polygons, are points, lines, and angles similarly situated.

ON SIMILAR POLYGONS.

PROPOSITION IV. THEOREM.

279. *Two triangles which are mutually equiangular are similar.*



In the △ ABC and $A'B'C'$ let ∠ A, B, C be equal to ∠ A', B', C' respectively.

We are to prove $AB : A'B' = AC : A'C' = BC : B'C'$.

Apply the △ $A'B'C'$ to the △ ABC,

so that ∠ A' shall coincide with ∠ A.

Then the △ $A'B'C'$ will take the position of △ AEH.

Now ∠ AEH (same as ∠ B') = ∠ B.

∴ EH is ∥ to BC, § 69

(*when two straight lines, lying in the same plane, are cut by a third straight line, if the ext. int. ∠ be equal the lines are parallel*).

∴ $AB : AE = AC : AH$, § 276

(*one side of a △ is to either part cut off by a line ∥ to the base, as the other side is to the corresponding part*).

Substitute for AE and AH their equals $A'B'$ and $A'C'$.

Then $AB : A'B' = AC : A'C'$.

In like manner we may prove

$AB : A'B' = BC : B'C'$.

∴ the two △ are similar. § 278

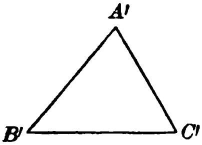
Q. E. D.

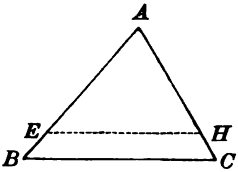
280. COR. 1. Two triangles are similar when two angles of the one are equal respectively to two angles of the other.

281. COR. 2. Two right triangles are similar when an acute angle of the one is equal to an acute angle of the other.

PROPOSITION V. THEOREM.

282. *Two triangles are similar when their homologous sides are proportional.*





In the triangles ABC and $A'B'C'$ let

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

We are to prove

$\angle$s A, B, and C equal respectively to $\angle$s A', B', and C'.

Take on AB, AE equal to $A'B'$,

and on AC, AH equal to $A'C'$. Draw EH.

$$\frac{AB}{A'B'} = \frac{AC}{A'C'}, \qquad \text{Hyp.}$$

Substitute in this equality, for $A'B'$ and $A'C'$ their equals AE and AH.

Then $$\frac{AB}{AE} = \frac{AC}{AH}.$$

$\therefore EH$ is $\parallel$ to BC, § 277

(*if a line divide two sides of a $\triangle$ proportionally, it is $\parallel$ to the third side*).

Now in the $\triangle$s ABC and AEH

$\angle ABC = \angle AEH$, § 70

(*being ext. int. angles*).

$\angle ACB = \angle AHE$, § 70

$\angle A = \angle A$. Iden.

$\therefore$ $\triangle$s ABC and AEH are similar, § 279

(*two mutually equiangular $\triangle$s are similar*).

$$\therefore \frac{AB}{BC} = \frac{AE}{EH}, \qquad \text{§ 278}$$

(*homologous sides of similar $\triangle$s are proportional*).

But $$\frac{AB}{BC} = \frac{A'B'}{B'C'},$$ § 278

$$\therefore \frac{AE}{EH} = \frac{A'B'}{B'C'}.$$ Ax. 1

Since $AE = A'B'$,

$EH = B'C'$. Cons.

Now in the $\triangle$s AEH and $A'B'C'$,

$$EH = B'C', \ AE = A'B', \text{ and } AH = A'C',$$

$$\therefore \triangle AEH = \triangle A'B'C',$$ § 108

(*having three sides of the one equal respectively to three sides of the other*).

But $\triangle AEH$ is similar to $\triangle ABC$.

$\therefore \triangle A'B'C'$ is similar to $\triangle ABC$.

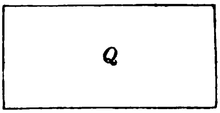
Q. E. D.

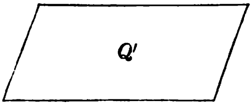
283. Scholium. The primary idea of similarity is *likeness of form;* and the two conditions necessary to similarity are:

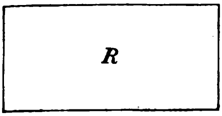
I. For every angle in one of the figures there must be an equal angle in the other, and

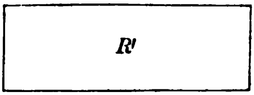
II. the homologous sides must be in proportion.

In the case of *triangles* either condition involves the other, but in the case of *other polygons*, it does not follow that if one condition exist the other does also.







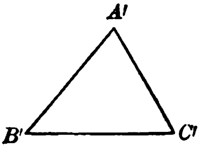


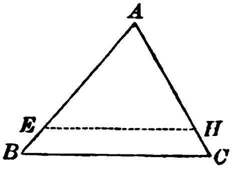
Thus in the quadrilaterals Q and Q', the homologous sides are proportional, but the homologous angles are not equal and the figures are not similar.

In the quadrilaterals R and R', the homologous angles are equal, but the sides are not proportional, and the figures are not similar.

PROPOSITION VI. THEOREM.

284. *Two triangles having an angle of the one equal to an angle of the other, and the including sides proportional, are similar.*





In the triangles ABC and $A'B'C'$ let

$$\angle A = \angle A', \text{ and } \frac{AB}{A'B'} = \frac{AC}{A'C'}.$$

We are to prove $\triangle$s ABC *and* $A'B'C'$ *similar.*

Apply the $\triangle\, A'B'C'$ to the $\triangle\, ABC$ so that $\angle A'$ shall coincide with $\angle A$.

Then the point B' will fall somewhere upon AB, as at E,

the point C' will fall somewhere upon AC, as at H, and $B'C'$ upon EH.

Now $$\frac{AB}{A'B'} = \frac{AC}{A'C'}.$$ Hyp.

Substitute for $A'B'$ and $A'C'$ their equals AE and AH.

Then $$\frac{AB}{AE} = \frac{AC}{AH}.$$

$\therefore$ the line EH divides the sides AB and AC proportionally;

$\therefore EH$ is $\parallel$ to BC, § 277

(*if a line divide two sides of a* $\triangle$ *proportionally, it is* $\parallel$ *to the third side*).

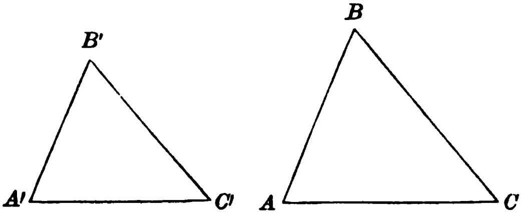
$\therefore$ the $\triangle$s ABC and AEH are mutually equiangular and similar.

$\therefore \triangle\, A'B'C'$ is similar to $\triangle\, ABC$.

Q. E. D

PROPOSITION VII. THEOREM.

285. *Two triangles which have their sides respectively parallel are similar.*



In the triangles ABC and $A'B'C'$ let AB, AC, and BC be parallel respectively to $A'B'$, $A'C'$, and $B'C'$.

We are to prove △ ABC *and* $A'B'C'$ *similar.*

The corresponding ∠s are either equal, § 77
(*two ∠s whose sides are ∥, two and two, and lie in the same direction, or opposite directions, from their vertices are equal*).

or supplements of each other, § 78
(*if two ∠s have two sides ∥ and lying in the same direction from their vertices, while the other two sides are ∥ and lie in opposite directions, the ∠s are supplements of each other*).

Hence we may make three suppositions:

1st. $A + A' = 2$ rt. ∠s, $B + B' = 2$ rt. ∠s, $C + C' = 2$ rt. ∠s.
2d. $A = A'$, $B + B' = 2$ rt. ∠s, $C + C' = 2$ rt. ∠s.
3d. $A = A'$, $B = B'$ $\therefore C = C'$.

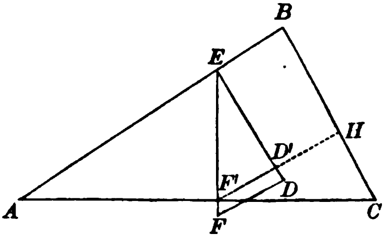
Since the sum of the ∠s of the two △ cannot exceed *four right angles*, the 3d supposition only is admissible. § 98

$\therefore$ the two △ ABC and $A'B'C'$ are similar, § 279
(*two mutually equiangular △ are similar*).

Q. E. D.

PROPOSITION VIII. THEOREM.

286. *Two triangles which have their sides respectively perpendicular to each other are similar.*



In the triangles EFD and BAC, let EF, FD and ED, be perpendicular respectively to AC, BC and AB.

We are to prove ⊿ *EFD and BAC similar.*

Place the $\triangle EFD$ so that its vertex E will fall on AB, and the side EF, ⊥ to AC, will cut AC at F'.

Draw $F'D'$ ∥ to FD, and prolong it to meet BC at H. In the quadrilateral $BED'H$, ∠s E and H are rt. ∠s.

$\therefore \angle B + \angle ED'H = 2$ rt. ∠s. § 158

But $\angle ED'F' + \angle ED'H = 2$ rt. ∠s. § 34

$\therefore \angle ED'F' = \angle B.$ Ax. 3.

Now $\angle C + \angle HF'C =$ rt. ∠, § 103

(*in a rt.* △ *the sum of the two acute* ∠s $=$ *a rt.* ∠);

and $\angle EF'D' + \angle HF'C =$ rt. ∠. Ax. 9.

$\therefore \angle EF'D' = \angle C.$ Ax. 3.

$\therefore$ ⊿ $EF'D'$ and BAC are similar. § 280

But $\triangle EFD$ is similar to $\triangle EF'D'$. § 279

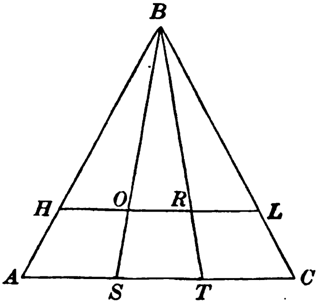
$\therefore$ ⊿ EFD and BAC are similar.

Q. E. D.

287. SCHOLIUM. When two triangles have their sides respectively parallel or perpendicular, the parallel sides, or the perpendicular sides, are homologous.

PROPOSITION IX. THEOREM.

288. *Lines drawn through the vertex of a triangle divide proportionally the base and its parallel.*



In the triangle ABC let HL be parallel to AC, and let BS and BT be lines drawn through its vertex to the base.

We are to prove

$$\frac{AS}{HO} = \frac{ST}{OR} = \frac{TC}{RL}.$$

△ BHO and BAS are similar, § 279
(*two △ which are mutually equiangular are similar*).

△ BOR and BST are similar, § 279

△ BRL and BTC are similar, § 279

$$\therefore \frac{AS}{HO} = \left(\frac{SB}{OB}\right) = \frac{ST}{OR} = \left(\frac{BT}{BR}\right) = \frac{TC}{RL}, \quad \S\ 278$$

(*homologous sides of similar △ are proportional*).

Q. E. D.

Ex. Show that, if three or more non-parallel straight lines divide two parallels proportionally, they pass through a common point.

PROPOSITION X. THEOREM.

289. *If in a right triangle a perpendicular be drawn from the vertex of the right angle to the hypotenuse:*

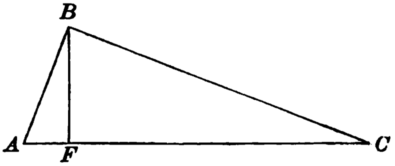
I. *It divides the triangle into two right triangles which are similar to the whole triangle, and also to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Each side of the right triangle is a mean proportional between the hypotenuse and its adjacent segment.*

IV. *The squares on the two sides of the right triangle have the same ratio as the adjacent segments of the hypotenuse.*

V. *The square on the hypotenuse has the same ratio to the square on either side as the hypotenuse has to the segment adjacent to that side.*



In the right triangle ABC, let BF be drawn from the vertex of the right angle B, perpendicular to the hypotenuse AC.

I. *We are to prove*

the △s ABF, ABC, and FBC similar.

In the rt. △s BAF and BAC,

the acute ∠ A is common.

∴ the △s are similar, § 281

(two rt. △s are similar when an acute ∠ of the one is equal to an acute ∠ of the other).

In the rt. △s BCF and BCA,

the acute ∠ C is common.

∴ the △s are similar. § 281

Now as the rt. △s ABF and CBF are both similar to ABC, by reason of the equality of their ∠s,

they are similar to each other.

II. *We are to prove* $AF : BF :: BF : FC$.

In the similar $\triangle$ ABF and CBF,

AF, the shortest side of the one,
: BF, the shortest side of the other,
: : BF, the medium side of the one,
: FC, the medium side of the other.

III. *We are to prove* $AC : AB :: AB : AF$.

In the similar $\triangle$ ABC and ABF,

AC, the longest side of the one,
: AB, the longest side of the other,
: : AB, the shortest side of the one,
: AF, the shortest side of the other.

Also in the similar $\triangle$ ABC and FBC,

AC, the longest side of the one,
: BC, the longest side of the other,
: : BC, the medium side of the one,
: FC, the medium side of the other.

IV. *We are to prove* $\dfrac{\overline{AB}^2}{\overline{BC}^2} = \dfrac{AF}{FC}$.

In the proportion $AC : AB :: AB : AF$,

$$\overline{AB}^2 = AC \times AF, \qquad \S\ 259$$

(*the product of the extremes is equal to the product of the means*).

and in the proportion $AC : BC :: BC : FC$,

$$\overline{BC}^2 = AC \times FC. \qquad \S\ 259$$

Dividing the one by the other,

$$\frac{\overline{AB}^2}{\overline{BC}^2} = \frac{AC \times AF}{AC \times FC}.$$

Cancel the common factor AC, and we have

$$\frac{\overline{AB}^2}{\overline{BC}^2} = \frac{AF}{FC}.$$

V. *We are to prove* $\dfrac{\overline{AC}^2}{\overline{AB}^2} = \dfrac{AC}{AF}$.

$$\overline{AC}^2 = AC \times AC.$$

$$\overline{AB}^2 = AC \times AF, \qquad \text{(Case III.)}$$

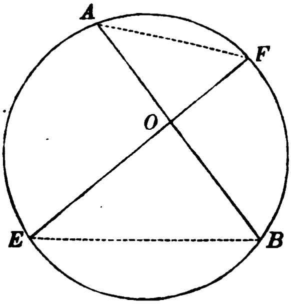
Divide one equation by the other;

then $$\frac{\overline{AC}^2}{\overline{AB}^2} = \frac{AC \times AC}{AC \times AF} = \frac{AC}{AF}.$$

Q. E. D.

PROPOSITION XI. THEOREM.

290. *If two chords intersect each other in a circle, their segments are reciprocally proportional.*



Let the two chords AB and EF intersect at the point O.

We are to prove $AO : EO :: OF : OB$.

Draw AF and EB.

In the △ AOF and EOB,

$\angle F = \angle B$, § 203

(*each being measured by ½ arc AE*).

$\angle A = \angle E$, § 203

(*each being measured by ½ arc FB*).

∴ the △ are similar. § 280

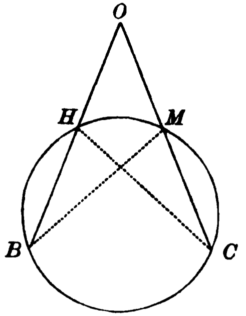
(*two △ are similar when two ∠ of the one are equal to two ∠ of the other*).

Whence AO, the medium side of the one, § 278
: EO, the medium side of the other,
: : OF, the shortest side of the one,
: OB, the shortest side of the other.

Q. E. D.

PROPOSITION XII. THEOREM.

291. *If from a point without a circle two secants be drawn, the whole secants and the parts without the circle are reciprocally proportional.*



Let OB and OC be two secants drawn from point O.

We are to prove $OB : OC :: OM : OH$.

Draw HC and MB.

In the $\triangle$s OHC and OMB

$\angle O$ is common,

$\angle B = \angle C$, § 203

(each being measured by $\frac{1}{2}$ arc HM).

$\therefore$ the two $\triangle$s are similar, § 280

(two $\triangle$s are similar when two $\angle$s of the one are equal to two $\angle$s of the other).

Whence OB, the longest side of the one, § 278

: OC, the longest side of the other,

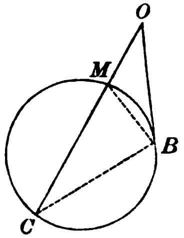
: : OM, the shortest side of the one,

: OH, the shortest side of the other.

Q. E. D.

PROPOSITION XIII. THEOREM.

292. *If from a point without a circle a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and the part without the circle.*



Let OB be a tangent and OC a secant drawn from the point O to the circle MBC.

We are to prove $OC : OB :: OB : OM$.

Draw BM and BC.

In the $\triangle$ OBM and OBC

$\angle O$ is common.

$\angle OBM$ is measured by $\frac{1}{2}$ arc MB, § 209

(being an $\angle$ formed by a tangent and a chord).

$\angle C$ is measured by $\frac{1}{2}$ arc BM, § 203

(being an inscribed $\angle$).

$\therefore \angle OBM = \angle C$.

$\therefore$ $\triangle$ OBC and OBM are similar, § 280

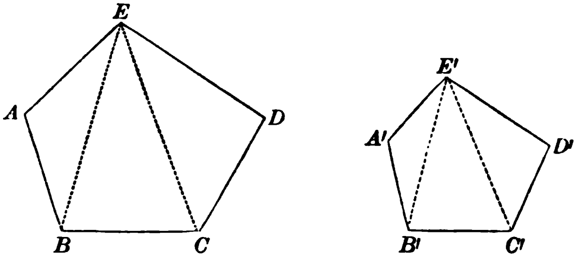
(having two $\angle$ of the one equal to two $\angle$ of the other).

Whence OC, the longest side of the one, § 278
: OB, the longest side of the other,
: : OB, the shortest side of the one,
: OM, the shortest side of the other.

Q. E. D.

PROPOSITION XIV. THEOREM.

293. *If two polygons be composed of the same number of triangles which are similar, each to each, and similarly placed, then the polygons are similar.*



In the two polygons $ABCDE$ and $A'B'C'D'E'$, let the triangles BAE, BEC, and CED be similar respectively to the triangles $B'A'E'$, $B'E'C'$, and $C'E'D'$.

We are to prove

the polygon $ABCDE$ similar to the polygon $A'B'C'D'E'$.

$$\angle A = \angle A', \qquad \text{§ 278}$$

(being homologous ∠s of similar △s).

$$\angle ABE = \angle A'B'E', \qquad \text{§ 278}$$

$$\angle EBC = \angle E'B'C', \qquad \text{§ 278}$$

Add the last two equalities.

Then $\angle ABE + \angle EBC = \angle A'B'E' + \angle E'B'C'$;

or, $\angle ABC = \angle A'B'C'$.

In like manner we may prove $\angle BCD = \angle B'C'D'$, etc.

∴ the two polygons are mutually equiangular.

Now $$\frac{AE}{A'E'} = \frac{AB}{A'B'} = \left(\frac{EB}{E'B'}\right) = \frac{BC}{B'C'} = \left(\frac{EC}{E'C'}\right) = \frac{CD}{C'D'} = \frac{ED}{E'D'},$$

(the homologous sides of similar △s are proportional).

∴ the homologous sides of the two polygons are proportional.

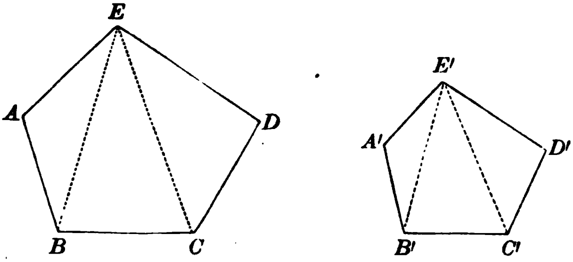
∴ the two polygons are similar, § 278

(having their homologous ∠s equal, and their homologous sides proportional).

Q. E. D.

PROPOSITION XV. THEOREM.

294. *If two polygons be similar, they are composed of the same number of triangles, which are similar and similarly placed.*



Let the polygons $ABCDE$ and $A'B'C'D'E'$ be similar.

From two homologous vertices, as E and E',

draw diagonals EB, EC, and $E'B'$, $E'C'$.

We are to prove $\triangle$s AEB, EBC, ECD

similar respectively to $\triangle$s $A'E'B'$, $E'B'C'$, $E'C'D'$.

In the $\triangle$s AEB and $A'E'B'$,

$$\angle A = \angle A', \qquad \S\ 278$$

(being homologous $\angle$s *of similar polygons).*

$$\frac{AE}{A'E'} = \frac{AB}{A'B'}, \qquad \S\ 278$$

(being homologous sides of similar polygons).

$\therefore$ $\triangle$s AEB and $A'E'B'$ are similar, § 284

(having an $\angle$ *of the one equal to an* $\angle$ *of the other, and the including sides proportional).*

Also, $\angle ABC = \angle A'B'C'$,

(being homologous $\angle$s *of similar polygons).*

$$\angle ABE = \angle A'B'E',$$

(being homologous $\angle$s *of similar* $\triangle$s*).*

$$\therefore \angle ABC - \angle ABE = \angle A'B'C' - \angle A'B'E'.$$

That is $\angle EBC = \angle E'B'C'$.

Now $$\frac{EB}{E'B'} = \frac{AB}{A'B'}, \qquad \S\ 278$$

(being homologous sides of similar $\triangle$*);*

also $$\frac{BC}{B'C'} = \frac{AB}{A'B'},$$

(being homologous sides of similar polygons).

$$\therefore \frac{EB}{E'B'} = \frac{BC}{B'C'}, \qquad \text{Ax. } 1$$

$\therefore \triangle\, EBC$ and $E'B'C'$ are similar, § 284

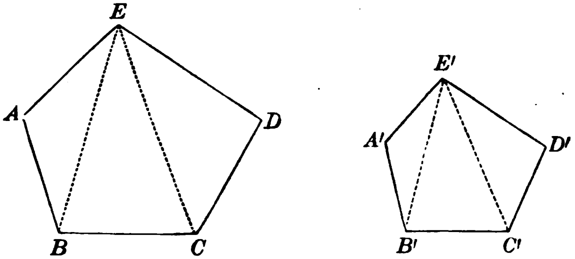
(having an $\angle$ *of the one equal to an* $\angle$ *of the other, and the including sides proportional).*

In like manner we may prove $\triangle\, ECD$ similar to $\triangle\, E'C'D'$.

Q. E. D.

PROPOSITION XVI. THEOREM.

295. *The perimeters of two similar polygons have the same ratio as any two homologous sides.*



Let the two similar polygons be $ABCDE$ and $A'B'C'D'E'$, and let P and P' represent their perimeters.

We are to prove $P : P' :: AB : A'B'$.

$AB : A'B' :: BC : B'C' :: CD : C'D'$ etc. § 278

(the homologous sides of similar polygons are proportional).

$\therefore AB + BC$, etc. $: A'B' + B'C'$, etc. $:: AB : A'B'$, § 266

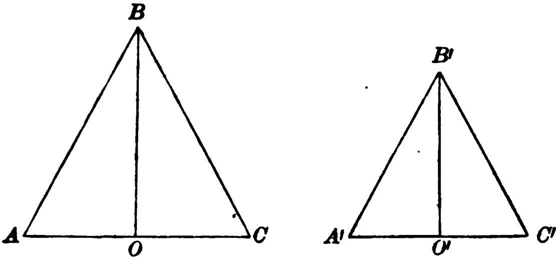
(in a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent).

That is $P : P' :: AB : A'B'$.

Q. E. D.

PROPOSITION XVII. THEOREM.

296. *The homologous altitudes of two similar triangles have the same ratio as any two homologous sides.*



In the two similar triangles ABC and $A'B'C'$, let the altitudes be BO and $B'O'$.

We are to prove $\frac{BO}{B'O'} = \frac{AB}{A'B'}$.

In the rt. ▲ BOA and $B'O'A'$,

$\angle A = \angle A'$ § 278

(being homologous ∠ *of the similar ▲ ABC and $A'B'C'$).*

$\therefore \triangle BOA$ and $\triangle B'O'A'$ are similar, § 281

(two rt. ▲ having an acute ∠ of the one equal to an acute ∠ of the other are similar).

$\therefore$ their homologous sides give the proportion

$$\frac{BO}{B'O'} = \frac{AB}{A'B'}.$$

Q. E. D

297. COR. 1. The homologous altitudes of similar triangles have the same ratio as their homologous bases.

In the similar △ $A B C$ and $A' B' C'$,

$$\frac{A C}{A' C'} = \frac{A B}{A' B'}, \qquad \S\ 278$$

(*the homologous sides of similar* △ *are proportional*).

And in the similar △ $B O A$ and $B' O' A'$,

$$\frac{B O}{B' O'} = \frac{A B}{A' B'}, \qquad \S\ 296$$

$$\therefore \frac{B O}{B' O'} = \frac{A C}{A' C'}, \qquad \text{Ax. 1}$$

298. Cor. 2. **The homologous altitudes of similar triangles have the same ratio as their perimeters.**

Denote the perimeter of the first by P, and that of the second by P'.

Then $$\frac{P}{P'} = \frac{A B}{A' B'}, \qquad \S\ 295$$

(*the perimeters of two similar polygons have the same ratio as any two homologous sides*).

But $$\frac{B O}{B' O'} = \frac{A B}{A' B'}; \qquad \S\ 296$$

$$\therefore \frac{B O}{B' O'} = \frac{P}{P'}. \qquad \text{Ax. 1}$$

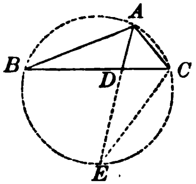
Ex. 1. If any two straight lines be cut by parallel lines, show that the corresponding segments are proportional.

2. If the four sides of any quadrilateral be bisected, show that the lines joining the points of bisection will form a parallelogram.

3. Two circles intersect; the line $A H K B$ joining their centres A, B, meets them in H, K. On $A B$ is described an equilateral triangle $A B C$, whose sides $B C$, $A C$, intersect the circles in F, E. $F E$ produced meets $B A$ produced in P. Show that as $P A$ is to $P K$ so is $C F$ to $C E$, and so also is $P H$ to $P B$.

PROPOSITION XVIII. THEOREM.

299. *In any triangle the product of two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle together with the square of the bisector.*



Let $\angle BAC$ of the $\triangle ABC$ be bisected by the straight line AD.

We are to prove $BA \times AC = BD \times DC + \overline{AD}^2$.

Describe the $\odot ABC$ about the $\triangle ABC$;

produce AD to meet the circumference in E, and draw EC.

Then in the $\triangle$s ABD and AEC,

$\angle BAD = \angle CAE$, Hyp.

$\angle B = \angle E$, § 203

(*each being measured by* $\frac{1}{2}$ *the arc* AC).

$\therefore$ $\triangle$s ABD and AEC are similar, § 280

(*two* $\triangle$s *are similar when two* $\angle$s *of the one are equal respectively to two* $\angle$s *of the other*).

Whence BA, the longest side of the one,

: EA, the longest side of the other,

: : AD, the shortest side of the one,

: AC, the shortest side of the other;

or, $$\frac{BA}{EA} = \frac{AD}{AC},$$ § 278

(*homologous sides of similar* $\triangle$s *are proportional*).

$\therefore BA \times AC = EA \times AD.$

But $EA \times AD = (ED + AD)\,AD,$

$\therefore BA \times AC = ED \times AD + \overline{AD}^2.$

But $ED \times AD = BD \times DC,$ § 290

(*the segments of two chords in a* $\odot$ *which intersect each other are reciprocally proportional*).

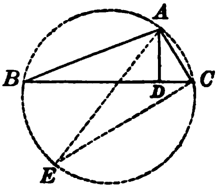
Substitute in the above equality $BD \times DC$ for $ED \times AD$,

then $BA \times AC = BD \times DC + \overline{AD}^2.$

Q. E. D.

PROPOSITION XIX. THEOREM.

300. *In any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle by the perpendicular let fall upon the third side from the vertex of the opposite angle.*



Let ABC be a triangle, and AD the perpendicular from A to BC.

Describe the circumference ABC about the $\triangle ABC$.

Draw the diameter AE, and draw EC.

We are to prove $BA \times AC = EA \times AD$.

In the ⚠ ABD and AEC

$\angle BDA$ is a rt. $\angle$, Cons.

$\angle ECA$ is a rt. $\angle$, § 204
(being inscribed in a semicircle).

$\therefore \angle BDA = \angle ECA$.

$\angle B = \angle E$, § 203
(each being measured by ½ the arc AC).

$\therefore$ ⚠ ABD and AEC are similar, § 281
(two rt. ⚠ having an acute $\angle$ of the one equal to an acute $\angle$ of the other are similar).

Whence BA, the longest side of the one,
: EA, the longest side of the other,
: : AD, the shortest side of the one,
: AC, the shortest side of the other;

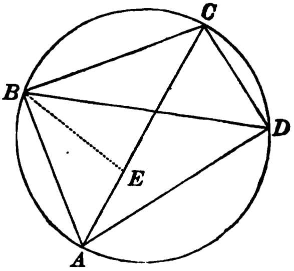
or,
$$\frac{BA}{EA} = \frac{AD}{AC}. \qquad \text{§ 278}$$

$\therefore BA \times AC = EA \times AD$.

Q. E. D.

PROPOSITION XX. THEOREM.

301. *The product of the two diagonals of a quadrilateral inscribed in a circle is equal to the sum of the products of its opposite sides.*



Let $ABCD$ be any quadrilateral inscribed in a circle, AC and BD its diagonals.

We are to prove $BD \times AC = AB \times CD + AD \times BC$.

Construct $\angle ABE = \angle DBC$,

and add to each $\angle EBD$.

Then in the $\triangle$ ABD and BCE,

$\angle ABD = \angle CBE$, Ax. 2

and $\angle BDA = \angle BCE$, § 203

(each being measured by $\frac{1}{2}$ the arc AB).

$\therefore$ $\triangle$ ABD and BCE, are similar, § 280

(two $\triangle$ are similar when two $\angle$ of the one are equal respectively to two $\angle$ of the other).

Whence AD, the medium side of the one,

: CE, the medium side of the other,

: : BD, the longest side of the one,

: BC, the longest side of the other,

or, $$\frac{AD}{CE} = \frac{BD}{BC},$$ § 278

(the homologous sides of similar △ are proportional).

$$\therefore BD \times CE = AD \times BC.$$

Again, in the △ ABE and BCD,

$$\angle ABE = \angle DBC,$$ Cons.

and $$\angle BAE = \angle BDC,$$ § 203

(each being measured by ½ of the arc BC).

∴ △ ABE and BCD are similar, § 280

(two △ are similar when two ∠ of the one are equal respectively to two ∠ of the other).

Whence AB, the longest side of the one,

: BD, the longest side of the other,

: : AE, the shortest side of the one,

: CD, the shortest side of the other.

or, $$\frac{AB}{BD} = \frac{AE}{CD},$$ § 278

(the homologous sides of similar △ are proportional).

$$\therefore BD \times AE = AB \times CD.$$

But $$BD \times CE = AD \times BC.$$

Adding these two equalities,

$$BD\,(AE + CE) = AB \times CD + AD \times BC,$$

or $$BD \times AC = AB \times CD + AD \times BC.$$

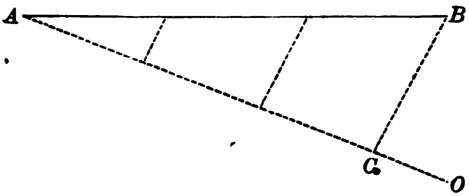
Q. E. D.

Ex. If two circles are tangent internally, show that chords of the greater, drawn from the point of tangency, are divided proportionally by the circumference of the less.

ON CONSTRUCTIONS.

PROPOSITION XXI. PROBLEM.

302. *To divide a given straight line into equal parts.*



Let $A B$ be the given straight line.

It is required to divide $A B$ into equal parts.

From A draw the indefinite line $A O$.

Take any convenient length, and apply it to $A O$ as many times as the line $A B$ is to be divided into parts.

From the last point thus found on $A O$, as C, draw $C B$.

Through the several points of division on $A O$ draw lines ‖ to $C B$.

These lines divide $A B$ into equal parts, § 274

(*if a series of* ‖*s intersecting any two straight lines, intercept equal parts on one of these lines, they intercept equal parts on the other also*).

Q. E. F.

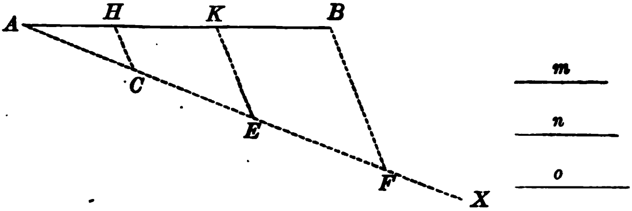
Ex. To draw a common tangent to two given circles.

I. When the common tangent is *exterior*.

II. When the common tangent is *interior*.

PROPOSITION XXII. PROBLEM.

303. *To divide a given straight line into parts proportional to any number of given lines.*



Let AB, m, n, and o be given straight lines.

It is required to divide AB into parts proportional to the given lines m, n, and o.

Draw the indefinite line AX.

On AX take $AC = m$,

$CE = n$,

and $EF = o$.

Draw FB. From E and C draw EK and CH ∥ to FB.

K and H are the division points required.

For $$\left(\frac{AK}{AE}\right) = \frac{AH}{AC} = \frac{HK}{CE} = \frac{KB}{EF},$$ § 275

(*a line drawn through two sides of a △ ∥ to the third side divides those sides proportionally*).

$$\therefore AH : HK : KB :: AC : CE : EF.$$

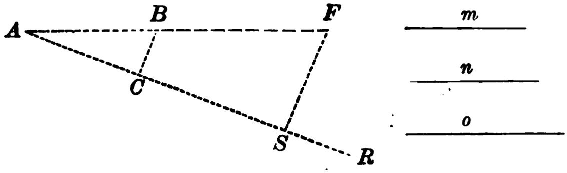
Substitute m, n, and o for their equals AC, CE, and EF.

Then $AH : HK : KB :: m : n : o.$

Q. E. F.

PROPOSITION XXIII. PROBLEM.

304. *To find a fourth proportional to three given straight lines.*



Let the three given lines be *m*, *n*, ***and*** *o*.

It is required to find a fourth proportional to m, n, and o.

Take AB equal to n.

Draw the indefinite line AR, making any convenient ∠ with AB.

On AR take $AC = m$, and $CS = o$.

Draw CB.

From S draw $SF \parallel$ to CB, to meet AB produced at F.

BF is the fourth proportional required.

For, $AC : AB :: CS : BF$, § 275

(*a line drawn through two sides of a* △ ∥ *to the third side divides those sides proportionally*).

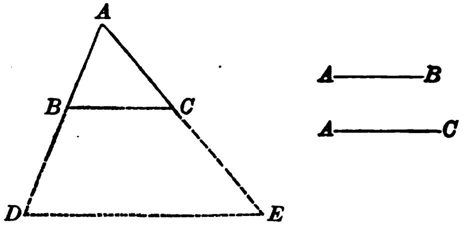
Substitute m, n, and o for their equals AC, AB, and CS.

Then $m : n :: o : BF$.

Q. E. F.

PROPOSITION XXIV. PROBLEM.

305. *To find a third proportional to two given straight lines.*



Let $A\,B$ and $A\,C$ be the two given straight lines.

It is required to find a third proportional to $A\,B$ and $A\,C$.

Place $A\,B$ and $A\,C$ so as to contain any convenient ∠.

Produce $A\,B$ to D, making $B\,D = A\,C$.

Join $B\,C$.

Through D draw $D\,E$ ∥ to $B\,C$ to meet $A\,C$ produced at E.

$C\,E$ is a third proportional to $A\,B$ and $A\,C$. § 251

For, $$\frac{A\,B}{B\,D} = \frac{A\,C}{C\,E},$$ § 275

(*a line drawn through two sides of a △ ∥ to the third side divides those sides proportionally*).

Substitute, in the above equality, $A\,C$ for its equal $B\,D$;

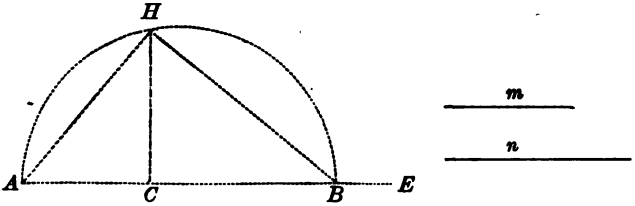
Then $$\frac{A\,B}{A\,C} = \frac{A\,C}{C\,E},$$

or, $$A\,B : A\,C :: A\,C : C\,E.$$

Q. E. F.

PROPOSITION XXV. PROBLEM.

306. *To find a mean proportional between two given lines.*



Let the two given lines be m and n.

It is required to find a mean proportional between m and n.

On the straight line AE

take $AC = m$, and $CB = n$.

On AB as a diameter describe a semi-circumference.

At C erect the $\perp$ CH.

CH is a mean proportional between m and n.

Draw HB and HA.

The $\angle AHB$ is a rt. $\angle$, § 204

(*being inscribed in a semicircle*),

and HC is a $\perp$ let fall from the vertex of a rt. $\angle$ to the hypotenuse.

$\therefore AC : CH :: CH : CB,$ § 289

(*the $\perp$ let fall from the vertex of the rt. $\angle$ to the hypotenuse is a mean proportional between the segments of the hypotenuse*).

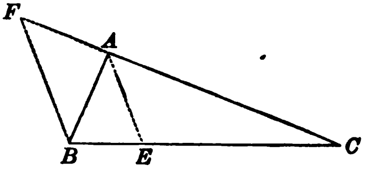
Substitute for AC and CB their equals m and n.

Then $m : CH :: CH : n.$ Q. E. F.

307. COROLLARY. *If from a point in the circumference a perpendicular be drawn to the diameter, and chords from the point to the extremities of the diameter, the perpendicular is a mean proportional between the segments of the diameter, and each chord is a mean proportional between its adjacent segment and the diameter.*

PROPOSITION XXVI. PROBLEM.

308. *To divide one side of a triangle into two parts proportional to the other two sides.*



Let ABC be the triangle.

It is required to divide the side BC into two such parts that the ratio of these two parts shall equal the ratio of the other two sides, AC and AB.

Produce CA to F, making $AF = AB$.

Draw FB.

From A draw $AE \parallel$ to FB.

E is the division point required.

For $$\frac{CA}{AF} = \frac{CE}{EB}. \qquad \S\ 275$$

(*a line drawn through two sides of a* △ ∥ *to the third side divides those sides proportionally*).

Substitute for AF its equal AB.

Then $$\frac{CA}{AB} = \frac{CE}{EB}.$$

Q. E. F.

309. COROLLARY. The line AE bisects the angle CAB.

For $$\angle F = \angle ABF, \qquad \S\ 112$$

(*being opposite equal sides*).

$$\angle F = \angle CAE, \qquad \S\ 70$$

(*being ext.-int.* $\angle$s).

$$\angle ABF = \angle BAE, \qquad \S\ 68$$

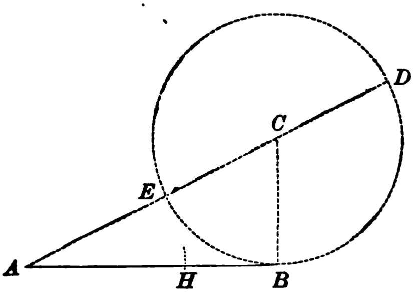
(*being alt.-int.* $\angle$s).

$$\therefore \angle CAE = \angle BAE. \qquad \text{Ax. } 1$$

310. DEF. A straight line is said to be divided *in extreme and mean ratio*, when the whole line is to the greater segment as the greater segment is to the less.

PROPOSITION XXVII. PROBLEM.

311. *To divide a given line in extreme and mean ratio.*



Let AB be the given line.

It is required to divide AB in extreme and mean ratio.

At B erect a $\perp$ BC, equal to one-half of AB.

From C as a centre, with a radius equal to CB, describe a ⊙.

Since AB is $\perp$ to the radius CB at its extremity, it is tangent to the circle.

Through C draw AD, meeting the circumference in E and D.

On AB take $AH = AE$.

H is the division point of AB required.

For $AD : AB :: AB : AE$, § 292

(*if from a point without the circumference a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and the part without the circumference*).

Then $AD - AB : AB :: AB - AE : AE$. § 265

Since $AB = 2\,CB$, Cons.

and $ED = 2\,CB$,

(the diameter of a ⊙ being twice the radius),

$AB = ED$. Ax. 1

$\therefore AD - AB = AD - ED = AE$.

But $AE = AH$, Cons.

$\therefore AD - AB = AH$. Ax. 1

Also $AB - AE = AB - AH = HB$.

Substitute these equivalents in the last proportion.

Then $AH : AB :: HB : AH$.

Whence, by inversion, $AB : AH :: AH : HB$. § 263

$\therefore AB$ is divided at H in extreme and mean ratio.

Q. E. F.

REMARK. AB is said to be divided at H, *internally*, in extreme and mean ratio. If BA be produced to H', making AH' equal to AD, AB is said to be divided at H', *externally*, in extreme and mean ratio.

Prove $AB : AH' :: AH' : H'B$.

When a line is divided internally and externally in the *same ratio*, it is said to be divided *harmonically*.

Thus AB

is divided harmonically at C and D, if $CA : CB :: DA : DB$; that is, if the ratio of the distances of C from A and B is equal to the ratio of the distances of D from A and B.

This proportion taken by alternation gives:

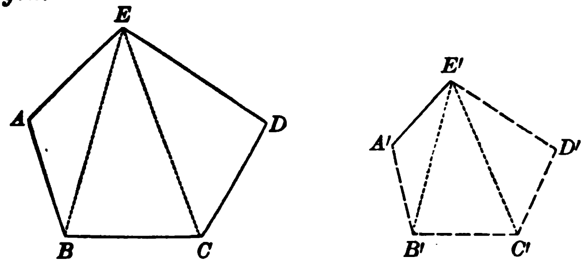
$AC : AD :: BC : BD$; that is, CD is divided harmonically at the points B and A. The four points A, B, C, D, are called *harmonic points*; and the two pairs A, B, and C, D, are called *conjugate points*.

Ex. 1. To divide a given line harmonically in a given ratio.

2. To find the locus of all the points whose distances from two given points are in a given ratio.

PROPOSITION XXVIII. PROBLEM.

312. *Upon a given line homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.*



Let $A'E'$ be the given line, homologous to AE of the given polygon $ABCDE$.

It is required to construct on $A'E'$ a polygon similar to the given polygon.

From E draw the diagonals EB and EC.

From E' draw $E'B'$, making $\angle A'E'B' = \angle AEB$.

Also from A' draw $A'B'$, making $\angle B'A'E' = \angle BAE$,

and meeting $E'B'$ at B'.

The two $\triangle$s ABE and $A'B'E'$ are similar, § 280

(*two $\triangle$s are similar if they have two $\angle$s of the one equal respectively to two $\angle$s of the other*).

Also from E' draw $E'C'$, making $\angle B'E'C' = \angle BEC$.

From B' draw $B'C'$, making $\angle E'B'C' = \angle EBC$,

and meeting $E'C'$ at C'.

Then the two $\triangle$s EBC and $E'B'C'$ are similar, § 280

(*two $\triangle$s are similar if they have two $\angle$s of the one equal respectively to two $\angle$s of the other*).

In like manner construct $\triangle E'C'D'$ similar to $\triangle ECD$.

Then the two polygons are similar, § 293

(*two polygons composed of the same number of $\triangle$s similar to each other and similarly placed, are similar*).

$\therefore$ $A'B'C'D'E'$ is the required polygon.

Q. E. F.

EXERCISES.

1. ABC is a triangle inscribed in a circle, and BD is drawn to meet the tangent to the circle at A in D, at an angle ABD equal to the angle ABC; show that AC is a fourth proportional to the lines BD, AD, AB.

2. Show that either of the sides of an isosceles triangle is a mean proportional between the base and the half of the segment of the base, produced if necessary, which is cut off by a straight line drawn from the vertex at right angles to the equal side.

3. AB is the diameter of a circle, D any point in the circumference, and C the middle point of the arc AD. If AC, AD, BC be joined and AD cut BC in E, show that the circle circumscribed about the triangle AEB will touch AC and its diameter will be a third proportional to BC and AB.

4. From the obtuse angle of a triangle draw a line to the base, which shall be a mean proportional between the segments into which it divides the base.

5. Find the point in the base produced of a right triangle, from which the line drawn to the angle opposite to the base shall have the same ratio to the base produced which the perpendicular has to the base itself.

6. A line touching two circles cuts another line joining their centres; show that the segments of the latter will be to each other as the diameters of the circles.

7. Required the locus of the middle points of all the chords of a circle which pass through a fixed point.

8. O is a fixed point from which any straight line is drawn meeting a fixed straight line at P; in OP a point Q is taken such that OQ is to OP in a fixed ratio. Determine the locus of Q.

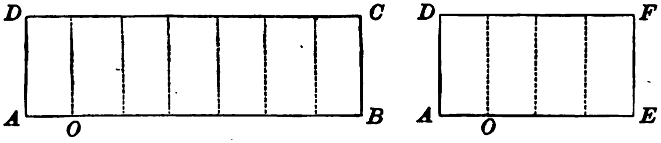
9. O is a fixed point from which any straight line is drawn meeting the circumference of a fixed circle at P; in OP a point Q is taken such that OQ is to OP in a fixed ratio. Determine the locus of Q.

BOOK IV.

COMPARISON AND MEASUREMENT OF THE SURFACES OF POLYGONS.

PROPOSITION I. THEOREM.

313. *Two rectangles having equal altitudes are to each other as their bases.*



Let the two rectangles be AC and AF, having the the same altitude AD.

We are to prove $\dfrac{\text{rect. } AC}{\text{rect. } AF} = \dfrac{AB}{AE}$.

CASE I. — *When AB and AE are commensurable.*

Find a common divisor of the bases AB and AE, as AO.

Suppose AO to be contained in AB seven times and in AE four times.

Then $$\frac{AB}{AE} = \frac{7}{4}.$$

At the several points of division on AB and AE erect $\perp$s.

The rect. AC will be divided into seven rectangles,

and rect. AF will be divided into four rectangles.

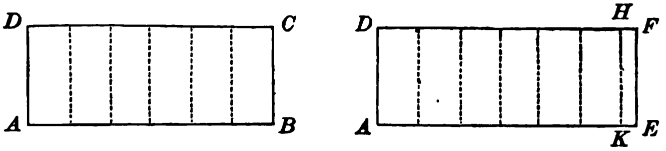
These rectangles are all equal, for they may be applied to each other and will coincide throughout.

$$\therefore \frac{\text{rect } AC}{\text{rect } AF} = \frac{7}{4}.$$

But $$\frac{AB}{AE} = \frac{7}{4}.$$

$$\therefore \frac{\text{rect } AC}{\text{rect } AF} = \frac{AB}{AE}.$$

CASE II. — *When AB and AE are incommensurable.*



Divide AB into any number of equal parts, and apply one of these parts to AE as often as it will be contained in AE.

Since AB and AE are incommensurable, a certain number of these parts will extend from A to a point K, leaving a remainder KE less than one of these parts.

Draw $KH \parallel$ to EF.

Since AB and AK are commensurable,

$$\frac{\text{rect. } AH}{\text{rect. } AC} = \frac{AK}{AB}, \qquad \text{Case 1}$$

Suppose the number of parts into which AB is divided to be continually increased, the length of each part will become less and less, and the point K will approach nearer and nearer to E.

The limit of AK will be AE, and the limit of rect. AH will be rect. AF.

$$\therefore \text{ the limit of } \frac{AK}{AB} \text{ will be } \frac{AE}{AB},$$

$$\text{and the limit of } \frac{\text{rect. } AH}{\text{rect. } AC} \text{ will be } \frac{\text{rect. } AF}{\text{rect. } AC}.$$

Now the variables $\frac{AK}{AB}$ and $\frac{\text{rect. } AH}{\text{rect. } AC}$ are always equal however near they approach their limits;

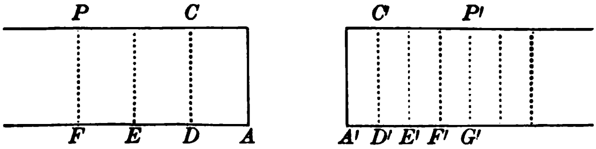
$$\therefore \text{ their limits are equal, namely, } \frac{\text{rect. } AF}{\text{rect. } AC} = \frac{AE}{AB}, \qquad \S\ 199$$

Q. E. D.

314. COROLLARY. Two rectangles having equal bases are to each other as their altitudes. By considering the bases of these two rectangles AD and AD, the altitudes will be AB and AE. But we have just shown that these two rectangles are to each other as AB is to AE. Hence two rectangles, with the same base, or equal bases, are to each other as their altitudes.

Another Demonstration.

Let $A C$ and $A' C'$ be two rectangles of equal altitudes.



We are to prove $\dfrac{\text{rect. } A C}{\text{rect. } A' C'} = \dfrac{A D}{A' D'}$.

Let b and b', S and S' stand for the bases and areas of these rectangles respectively.

Take $A D$, $D E$, $E F$ . . . . m in number and all equal,

and $A' D'$, $D' E'$, $E' F'$, $F' G'$ . . . . n in number and all equal.

Complete the rectangles as in the figure.

Then base $A F = m b$,

and base $A' G' = n b'$;

rect. $A P = m S$,

and rect. $A' P' = n S'$.

Now we can prove by superposition, that if $A F$ be $> A' G'$, rect. $A P$ will be $>$ rect. $A' P'$; and if equal, equal; and if less, less.

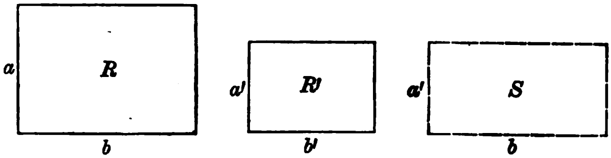
That is, if $m b$ be $> n b'$, $m S$ is $> n S'$; and if equal, equal; and if less, less.

Hence, $b : b' :: S : S'$, Euclid's Def., § 272

Q. E. D.

PROPOSITION II. THEOREM.

315. *Two rectangles are to each other as the product of their bases by their altitudes.*



Let R and R' be two rectangles, having for their bases b and b', and for their altitudes a and a'.

We are to prove $\dfrac{R}{R'} = \dfrac{a \times b}{a' \times b'}$.

Construct the rectangle S, with its base the same as that of R and its altitude the same as that of R'.

Then $\dfrac{R}{S} = \dfrac{a}{a'}$, § 314

(rectangles having the same base are to each other as their altitudes);

and $\dfrac{S}{R'} = \dfrac{b}{b'}$, § 313

(rectangles having the same altitude are to each other as their bases).

By multiplying these two equalities together

$$\frac{R}{R'} = \frac{a \times b}{a' \times b'}.$$

Q. E. D.

316. DEF. The *Area* of a surface is the ratio of that surface to another surface assumed as the unit of measure.

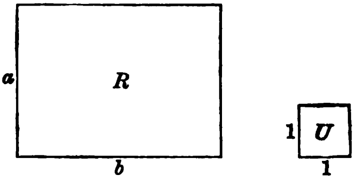
317. DEF. The *Unit of measure* (except the *acre*) is a square a side of which is some linear unit; as a square inch, etc.

318. DEF. Equivalent figures are figures which have equal areas.

REM. In comparing the areas of equivalent figures the symbol (=) is to be read "equal in area."

PROPOSITION III. THEOREM.

319. *The area of a rectangle is equal to the product of its base and altitude.*



Let R be the rectangle, b the base, and a the altitude; and let U be a square whose side is the linear unit.

We are to prove *the area of* $R = a \times b$.

$$\frac{R}{U} = \frac{a \times b}{1 \times 1}, \qquad \S\ 315$$

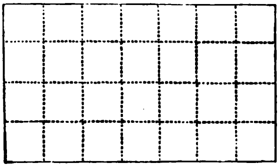
(two rectangles are to each other as the product of their bases and altitudes).

But $\frac{R}{U}$ is the area of R, § 316

$\therefore$ the area of $R = a \times b$.

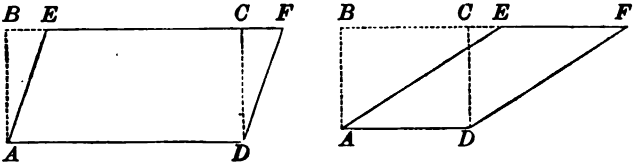
Q. E. D.

320. SCHOLIUM. When the base and altitude are exactly divisible by the linear unit, this proposition is rendered evident by dividing the figure into squares, each equal to the unit of measure. Thus, if the base contain seven linear units, and the altitude four, the figure may be divided into twenty-eight squares, each equal to the unit of measure; and the area of the figure equals 7 × 4.



PROPOSITION IV. THEOREM.

321. *The area of a parallelogram is equal to the product of its base and altitude.*



Let $AEFD$ be a parallelogram, AD its base, and CD its altitude.

We are to prove *the area of the* ▱ $AEFD = AD \times CD$.

From A draw $AB \parallel$ to DC to meet FE produced.

Then the figure $ABCD$ will be a rectangle, with the same base and altitude as the ▱ $AEFD$.

In the rt. △ ABE and CDF,

$AB = CD$, § 126
(being opposite sides of a rectangle).

and $AE = DF$, § 134
(being opposite sides of a ▱ *)*;

$\therefore \triangle ABE = \triangle CDF$, § 109
(two rt. △ *are equal, when the hypotenuse and a side of the one are equal respectively to the hypotenuse and a side of the other).*

Take away the $\triangle CDF$ and we have left the rect. $ABCD$.

Take away the $\triangle ABE$ and we have left the ▱ $AEFD$.

$\therefore$ rect. $ABCD =$ ▱ $AEFD$. Ax. 3

But the area of the rect. $ABCD = AD \times CD$, § 319
(the area of a rectangle equals the product of its base and altitude).

$\therefore$ the area of the ▱ $AEFD = AD \times CD$. Ax. 1

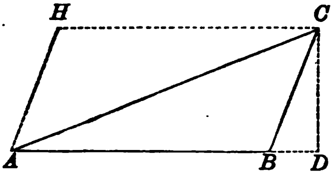
Q. E. D.

322. COROLLARY 1. Parallelograms having equal bases and equal altitudes are equivalent.

323. COR. 2. Parallelograms having equal bases are to each other as their altitudes; parallelograms having equal altitudes are to each other as their bases; and any two parallelograms are to each other as the products of their bases by their altitudes.

PROPOSITION V. THEOREM.

324. *The area of a triangle is equal to one-half of the product of its base by its altitude.*



Let ABC be a triangle, AB its base, and CD its altitude.

We are to prove the area of the $\triangle ABC = \frac{1}{2} AB \times CD$.

From C draw $CH \parallel$ to AB.

From A draw $AH \parallel$ to BC.

The figure $ABCH$ is a parallelogram, § 136
(having its opposite sides parallel),

and AC is its diagonal.

$\therefore \triangle ABC = \triangle AHC$, § 133
(the diagonal of a ▱ divides it into two equal △s).

The area of the ▱ $ABCH$ is equal to the product of its base by its altitude. § 321

$\therefore$ the area of one-half the ▱, or the $\triangle ABC$, is equal to one-half the product of its base by its altitude,

or, $\frac{1}{2} AB \times CD$.

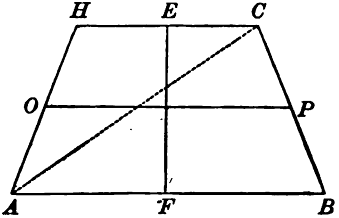
Q. E. D.

325. COROLLARY 1. Triangles having equal bases and equal altitudes are equivalent.

326. COR. 2. Triangles having equal bases are to each other as their altitudes; triangles having equal altitudes are to each other as their bases; any two triangles are to each other as the product of their bases by their altitudes.

PROPOSITION VI. THEOREM.

327. *The area of a trapezoid is equal to one-half the sum of the parallel sides multiplied by the altitude.*



Let $ABCH$ be a trapezoid, and EF the altitude.

We are to prove *area of* $ABCH = \frac{1}{2}(HC + AB)\,EF$.

Draw the diagonal AC.

Then the area of the $\triangle\ AHC = \frac{1}{2}\,HC \times EF$, § 324

(the area of a △ is equal to one-half of the product of its base by its altitude),

and the area of the $\triangle\ ABC = \frac{1}{2}\,AB \times EF$, § 324

$\therefore \triangle\ AHC + \triangle\ ABC$,

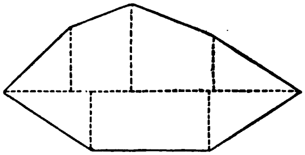
or, area of $ABCH = \frac{1}{2}(HC + AB)\,EF$.

Q. E. D.

328. COROLLARY. The area of a trapezoid is equal to the product of the line joining the middle points of the non-parallel sides multiplied by the altitude; for the line OP, joining the middle points of the non-parallel sides, is equal to $\frac{1}{2}(HC + AB)$. § 142

$\therefore$ by substituting OP for $\frac{1}{2}(HC + AB)$, we have,

the area of $ABCH = OP \times EF$.

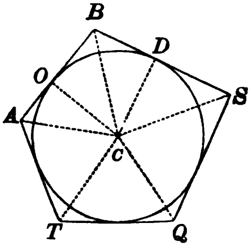


329. SCHOLIUM. The area of an irregular polygon may be found by dividing the polygon into triangles, and by finding the area of each of these triangles separately. But the method generally employed in practice is to draw the longest diagonal, and to let fall perpendiculars upon this diagonal from the other angular points of the polygon.

The polygon is thus divided into figures which are right triangles, rectangles, or trapezoids; and the areas of each of these figures may be readily found.

PROPOSITION VII. THEOREM.

330. *The area of a circumscribed polygon is equal to one-half the product of the perimeter by the radius of the inscribed circle.*



Let $ABSQ$, etc., be a circumscribed polygon, and C the centre of the inscribed circle.

Denote the perimeter of the polygon by P, and the radius of the inscribed circle by R.

We are to prove

the area of the circumscribed polygon $= \frac{1}{2} P \times R$.

Draw CA, CB, CS, etc.;

also draw CO, CD, etc., $\perp$ to AB, BS, etc.

The area of the $\triangle CAB = \frac{1}{2} AB \times CO$, § 324

(*the area of a* $\triangle$ *is equal to one-half the product of its base and altitude*).

The area of the $\triangle CBS = \frac{1}{2} BS \times CD$, § 324

$\therefore$ the area of the sum of all the $\triangle$ CAB, CBS, etc., $= \frac{1}{2}(AB + BS$, etc.$)\, CO$,

(*for* CO, CD, *etc., are equal, being radii of the same* ⊙).

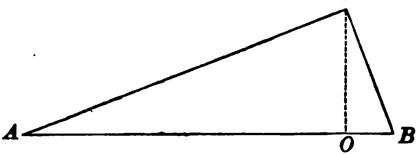
Substitute for $AB + BS + SQ$, etc., P, and for CO, R;

then the area of the circumscribed polygon $= \frac{1}{2} P \times R$.

Q. E. D.

PROPOSITION VIII. THEOREM.

331. *The sum of the squares described on the two sides of a right triangle is equivalent to the square described on the hypotenuse.*



Let ABC be a right triangle with its right angle at C.

We are to prove $\overline{AC}^2 + \overline{CB}^2 = \overline{AB}^2$

Draw $CO \perp$ to AB.

Then $\overline{AC}^2 = AO \times AB$, § 289

(*the square on a side of a rt.* △ *is equal to the product of the hypotenuse by the adjacent segment made by the* ⊥ *let fall from the vertex of the rt.* ∠) ;

and $\overline{BC}^2 = BO \times AB$, § 289

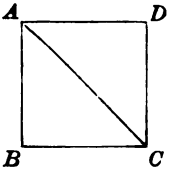
By adding, $\overline{AC}^2 + \overline{BC}^2 = (AO + BO)\,AB$,

$$= AB \times AB,$$

$$= \overline{AB}^2.$$

Q. E. D.

332. COROLLARY. The side and diagonal of a square are incommensurable.



Let $ABCD$ be a square, and AC the diagonal.

Then $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$.

or, $2\,\overline{AB}^2 = \overline{AC}^2$.

Divide both sides of the equation by $\overline{AB}^2$,

$$\frac{\overline{AC}^2}{\overline{AB}^2} = 2.$$

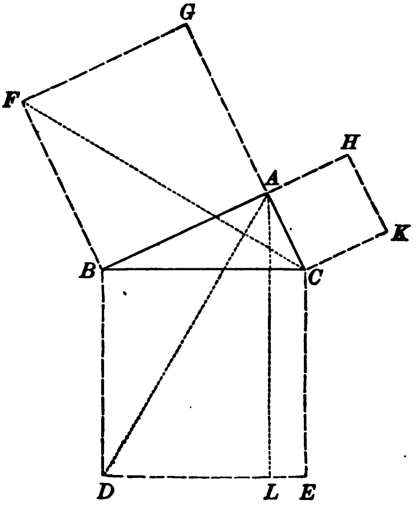
Extract the square root of both sides the equation,

then $$\frac{AC}{AB} = \sqrt{2}.$$

Since the square root of 2 is a number which cannot be exactly found, it follows that the diagonal and side of a square are two incommensurable lines.

ANOTHER DEMONSTRATION.

333. *The square described on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.*



Let ABC be a right △, having the right angle BAC.

We are to prove $\overline{BC}^2 = \overline{BA}^2 + \overline{AC}^2$.

On BC, CA, AB construct the squares BE, CH, AF.

Through A draw $AL \parallel$ to CE.

Draw AD and FC.

	$\angle BAC$ is a rt. $\angle$,	Hyp.
and	$\angle BAG$ is a rt. $\angle$,	Cons.
	$\therefore CAG$ is a straight line.	
Also	$\angle CAH$ is a rt. $\angle$,	Cons.
	$\therefore BAH$ is a straight line.	
Now	$\angle DBC = \angle FBA$, *(each being a rt. $\angle$).*	Cons.

Add to each the $\angle ABC$;

then $\angle ABD = \angle FBC$,

$\therefore \triangle ABD = \triangle FBC$. § 106

Now $\square BL$ is double $\triangle ABD$,

(being on the same base BD, and between the same ‖s, AL and BD),

and square AF is double $\triangle FBC$,

(being on the same base FB, and between the same ‖s, FB and GC);

$\therefore \square BL =$ square AF.

In like manner, by joining AE and BK, it may be proved that

$\square CL =$ square CH.

Now the square on $BC = \square BL + \square CL$,

$=$ square $AF +$ square CH,

$\therefore \overline{BC}^2 = \overline{BA}^2 + \overline{AC}^2$.

Q. E. D.

On Projection.

334. Def. *The Projection of a Point* upon a straight line of indefinite length is the foot of the perpendicular let fall from the point upon the line. Thus, the projection of the point C upon the line AB is the point P.

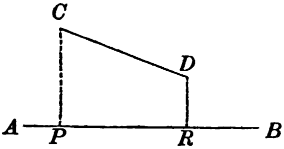


Fig. 1.

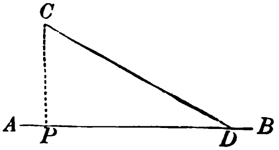


Fig. 2.

The Projection of a Finite Straight Line, as CD (Fig. 1), upon a straight line of indefinite length, as AB, is the part of the line AB intercepted between the perpendiculars CP and DR, let fall from the extremities of the line CD.

Thus the projection of the line CD upon the line AB is the line PR.

If one extremity of the line CD (Fig. 2) be in the line AB, the projection of the line CD upon the line AB is the part of the line AB between the point D and the foot of the perpendicular CP; that is, DP.

PROPOSITION IX. THEOREM.

335. *In any triangle, the square on the side opposite an acute angle is equivalent to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other upon that side.*

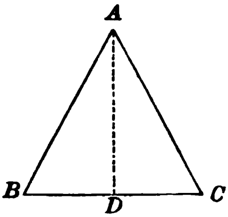


Fig. 1.

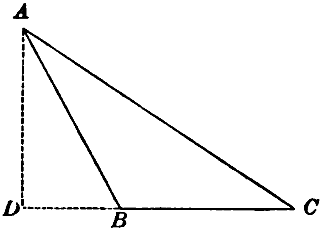


Fig. 2.

Let C be an acute angle of the triangle ABC, and DC the projection of AC upon BC.

We are to prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times DC.$

If D fall upon the base (Fig. 1),

$$DB = BC - DC;$$

If D fall upon the base produced (Fig. 2),

$$DB = DC - BC.$$

In either case $\overline{DB}^2 = \overline{BC}^2 + \overline{DC}^2 - 2\,BC \times DC.$

Add $\overline{AD}^2$ to both sides of the equality;

then, $\overline{AD}^2 + \overline{DB}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{DC}^2 - 2\,BC \times DC.$

But $\overline{AD}^2 + \overline{DB}^2 = \overline{AB}^2,$ § 331

(*the sum of the squares on two sides of a rt. △ is equivalent to the square on the hypotenuse*);

and $\overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2,$ § 331

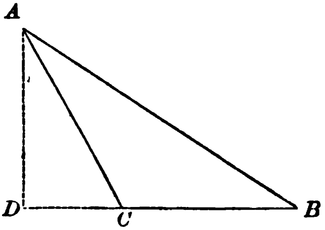
Substitute $\overline{AB}^2$ and $\overline{AC}^2$ for their equivalents in the above equality;

then, $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times DC.$

Q. E. D.

PROPOSITION X. THEOREM.

336. *In any obtuse triangle, the square on the side opposite the obtuse angle is equivalent to the sum of the squares of the other two sides increased by twice the product of one of those sides and the projection of the other on that side.*



Let C be the obtuse angle of the triangle ABC, and CD be the projection of AC upon BC produced.

We are to prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times DC$.

$$DB = BC + DC.$$

Squaring, $\overline{DB}^2 = \overline{BC}^2 + \overline{DC}^2 + 2\,BC \times DC$.

Add $\overline{AD}^2$ to both sides of the equality;

then, $\overline{AD}^2 + \overline{DB}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{DC}^2 + 2\,BC \times DC$.

But $\overline{AD}^2 + \overline{DB}^2 = \overline{AB}^2$, § 331

(*the sum of the squares on two sides of a rt. △ is equivalent to the square on the hypotenuse*);

and $\overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2$. § 331

Substitute $\overline{AB}^2$ and $\overline{AC}^2$ for their equivalents in the above equality;

then, $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times DC$.

Q. E. D.

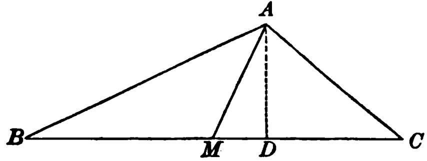
337. DEFINITION. A *Medial* line of a triangle is a straight line drawn from any vertex of the triangle to the middle point of the opposite side.

PROPOSITION XI. THEOREM.

338. *In any triangle, if a medial line be drawn from the vertex to the base:*

I. *The sum of the squares on the two sides is equivalent to twice the square on half the base, increased by twice the square on the medial line;*

II. *The difference of the squares on the two sides is equivalent to twice the product of the base by the projection of the medial line upon the base.*



In the triangle $A\,B\,C$ let $A\,M$ be the medial line and $M\,D$ the projection of $A\,M$ upon the base $B\,C$. Also let $A\,B$ be greater than $A\,C$.

We are to prove

$$\text{I.}\quad \overline{AB}^2 + \overline{AC}^2 = 2\,\overline{BM}^2 + 2\,\overline{AM}^2.$$

$$\text{II.}\quad \overline{AB}^2 - \overline{AC}^2 = 2\,BC \times MD.$$

Since $AB > AC$, the $\angle\, AMB$ will be obtuse and the $\angle\, AMC$ will be acute.

Then $\overline{AB}^2 = \overline{BM}^2 + \overline{AM}^2 + 2\,BM \times MD,$ § 336

(*in any obtuse △ the square on the side opposite the obtuse ∠ is equivalent to the sum of the squares on the other two sides increased by twice the product of one of those sides and the projection of the other on that side*);

and $\overline{AC}^2 = \overline{MC}^2 + \overline{AM}^2 - 2\,MC \times MD,$ § 335

(*in any △ the square on the side opposite an acute ∠ is equivalent to the sum of the squares on the other two sides, diminished by twice the product of one of those sides and the projection of the other upon that side*).

Add these two equalities, and observe that $BM = MC$.

Then $\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{BM}^2 + 2\,\overline{AM}^2.$

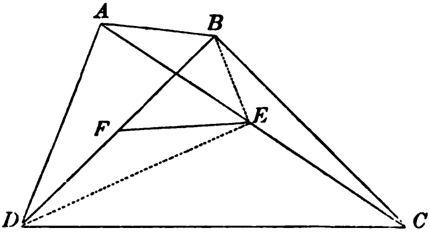
Subtract the second equality from the first.

Then $\overline{AB}^2 - \overline{AC}^2 = 2\,BC \times MD.$

Q. E. D.

PROPOSITION XII. THEOREM.

∂ 339. *The sum of the squares on the four sides of any quadrilateral is equivalent to the sum of the squares on the diagonals together with four times the square of the line joining the middle points of the diagonals.*



In the quadrilateral $ABCD$, let the diagonals be AC and BD, and FE the line joining the middle points of the diagonals.

We are to prove

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2.$$

Draw BE and DE.

Now $\overline{AB}^2 + \overline{BC}^2 = 2\left(\frac{AC}{2}\right)^2 + 2\,\overline{BE}^2$, § 338

(*the sum of the squares on the two sides of a △ is equivalent to twice the square on half the base increased by twice the square on the medial line to the base*),

and $\overline{CD}^2 + \overline{DA}^2 = 2\left(\frac{AC}{2}\right)^2 + 2\,\overline{DE}^2$. § 338

Adding these two equalities,

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\left(\frac{AC}{2}\right)^2 + 2\left(\overline{BE}^2 + \overline{DE}^2\right).$$

But $\overline{BE}^2 + \overline{DE}^2 = 2\left(\frac{BD}{2}\right)^2 + 2\,\overline{EF}^2$, § 338

(*the sum of the squares on the two sides of a △ is equivalent to twice the square on half the base increased by twice the square on the medial line to the base*).

Substitute in the above equality for $\left(\overline{BE}^2 + \overline{DE}^2\right)$ its equivalent;

then $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\left(\frac{AC}{2}\right)^2 + 4\left(\frac{BD}{2}\right)^2 + 4\,\overline{EF}^2$

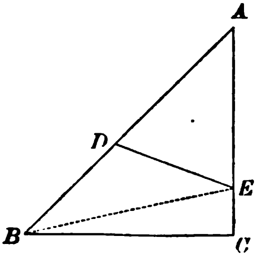
$$= \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2$$

Q. E. D.

340. COROLLARY. The sum of the squares on the four sides of a parallelogram is equivalent to the sum of the squares on the diagonals.

PROPOSITION XIII. THEOREM.

341. *Two triangles having an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Let the triangles ABC and ADE have the common angle A.

We are to prove $$\frac{\triangle ABC}{\triangle ADE} = \frac{AB \times AC}{AD \times AE}.$$

Draw BE.

Now $$\frac{\triangle ABC}{\triangle ABE} = \frac{AC}{AE},$$ § 326

(*△ having the same altitude are to each other as their bases*).

Also $$\frac{\triangle ABE}{\triangle ADE} = \frac{AB}{AD},$$ § 32

(*△ having the same altitude are to each other as their bases*).

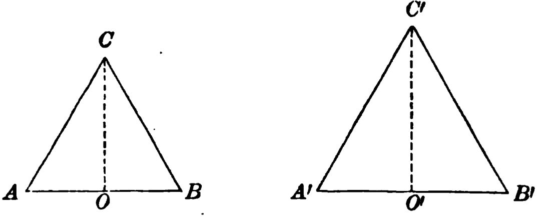
Multiply these equalities;

then $$\frac{\triangle ABC}{\triangle ADE} = \frac{AB \times AC}{AD \times AE}.$$

Q. E. D.

PROPOSITION XIV. THEOREM.

342. *Similar triangles are to each other as the squares on their homologous sides.*



Let the two triangles be ACB and $A'C'B'$.

We are to prove $\dfrac{\triangle ACB}{\triangle A'C'B'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

Draw the perpendiculars CO and $C'O'$.

Then $\dfrac{\triangle ACB}{\triangle A'C'B'} = \dfrac{AB \times CO}{A'B' \times C'O'} = \dfrac{AB}{A'B'} \times \dfrac{CO}{C'O'}$, § 326

(*two* $\triangle$s *are to each other as the products of their bases by their altitudes*).

But $\dfrac{AB}{A'B'} = \dfrac{CO}{C'O'}$, § 297

(*the homologous altitudes of similar* $\triangle$s *have the same ratio as their homologous bases*).

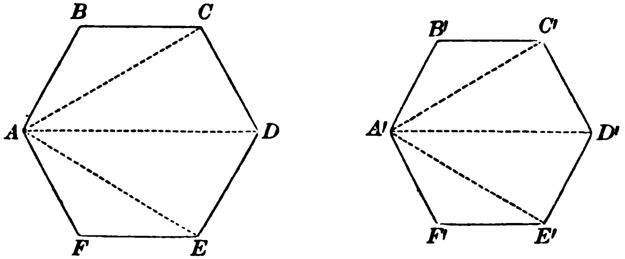
Substitute, in the above equality, for $\dfrac{CO}{C'O'}$ its equal $\dfrac{AB}{A'B'}$;

then $\dfrac{\triangle ACB}{\triangle A'C'B'} = \dfrac{AB}{A'B'} \times \dfrac{AB}{A'B'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

Q. E. D.

PROPOSITION XV. THEOREM.

343. *Two similar polygons are to each other as the squares on any two homologous sides.*



Let the two similar polygons be $A\,B\,C$, etc., and $A'\,B'\,C'$, etc.

We are to prove $\dfrac{A\,B\,C, \text{ etc.}}{A'\,B'\,C', \text{ etc.}} = \dfrac{\overline{A\,B}^2}{\overline{A'\,B'}^2}$.

From the homologous vertices A and A' draw diagonals.

Now $\dfrac{A\,B}{A'\,B'} = \dfrac{B\,C}{B'\,C'} = \dfrac{C\,D}{C'\,D'}$, etc.,

(*similar polygons have their homologous sides proportional*);

$\therefore$ by squaring, $\dfrac{\overline{A\,B}^2}{\overline{A'\,B'}^2} = \dfrac{\overline{B\,C}^2}{\overline{B'\,C'}^2} = \dfrac{\overline{C\,D}^2}{\overline{C'\,D'}^2}$, etc.

The $\triangle$s $A\,B\,C$, $A\,C\,D$, etc., are respectively similar to $A'\,B'\,C'$, $A'\,C'\,D'$, etc., § 294

(*two similar polygons are composed of the same number of* $\triangle$s *similar to each other and similarly placed*).

$\therefore \dfrac{\triangle\, A\,B\,C}{\triangle\, A'\,B'\,C'} = \dfrac{\overline{A\,B}^2}{\overline{A'\,B'}^2}$, § 342

(*similar* $\triangle$s *are to each other as the squares on their homologous sides*),

and $\dfrac{\triangle\, A\,C\,D}{\triangle\, A'\,C'\,D'} = \dfrac{\overline{C\,D}^2}{\overline{C'\,D'}^2}$. § 342

But $$\frac{\overline{CD}^2}{\overline{C'D'}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2},$$

$$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{\triangle ACD}{\triangle A'C'D'}.$$

In like manner we may prove that the ratio of any two of the similar ⊿s is the same as that of any other two.

$$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{\triangle ACD}{\triangle A'C'D'} = \frac{\triangle ADE}{\triangle A'D'E'} = \frac{\triangle AEF}{\triangle A'E'F'},$$

$$\therefore \frac{\triangle s\ ABC + ACD + ADE + AEF}{\triangle s\ A'B'C' + A'C'D' + A'D'E' + A'E'F'} = \frac{\triangle ABC}{\triangle A'B'C'},$$

(in a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent).

But $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}, \qquad \text{§ 342}$$

(similar ⊿s are to each other as the squares on their homologous sides);

$$\therefore \frac{\text{the polygon } ABC\text{, etc.}}{\text{the polygon } A'B'C'\text{, etc.}} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Q. E. D.

344. Corollary 1. Similar polygons are to each other as the squares on any two homologous lines.

345. Cor. 2. The homologous sides of two similar polygons have the same ratio as the square roots of their areas.

Let S and S' represent the areas of the two similar polygons ABC, etc., and $A'B'C'$, etc., respectively.

Then $$S : S' :: \overline{AB}^2 : \overline{A'B'}^2,$$

(similar polygons are to each other as the squares of their homologous sides).

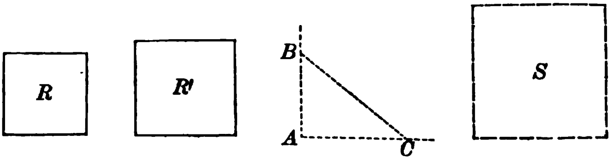
$$\therefore \quad \sqrt{S} : \sqrt{S'} :: AB : A'B', \qquad \text{§ 268}$$

or, $$AB : A'B' :: \sqrt{S} : \sqrt{S'}.$$

ON CONSTRUCTIONS.

PROPOSITION XVI. PROBLEM.

346. *To construct a square equivalent to the sum of two given squares.*



Let R and R' be two given squares.

It is required to construct a square $= R + R'$.

Construct the rt. $\angle A$.

Take AB equal to a side of R,

and AC equal to a side of R'.

Draw BC.

Then BC will be a side of the square required.

For $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$, § 331

(the square on the hypotenuse of a rt. $\triangle$ *is equivalent to the sum of the squares on the two sides).*

Construct the square S, having each of its sides equal to BC.

Substitute for $\overline{BC}^2$, $\overline{AB}^2$ and $\overline{AC}^2$, S, R, and R' respectively;

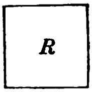
then $S = R + R'$.

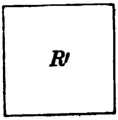
$\therefore$ S is the square required.

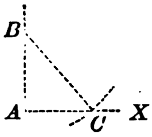
Q. E. F.

PROPOSITION XVII PROBLEM.

347. *To construct a square equivalent to the difference of two given squares.*







Let R be the smaller square and R' the larger.

It is required to construct a square $= R' - R$.

Construct the rt. $\angle A$.

Take AB equal to a side of R.

From B as a centre, with a radius equal to a side of R',

describe an arc cutting the line AX at C.

Then AC will be a side of the square required.

For draw BC.

$$\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2, \qquad \S\ 331$$

(*the sum of the squares on the two sides of a rt.* $\triangle$ *is equivalent to the square on the hypotenuse*).

By transposing, $\overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2$.

Construct the square S, having each of its sides equal to AC.

Substitute for $\overline{AC}^2$, $\overline{BC}^2$, and $\overline{AB}^2$, S, R', and R respectively;

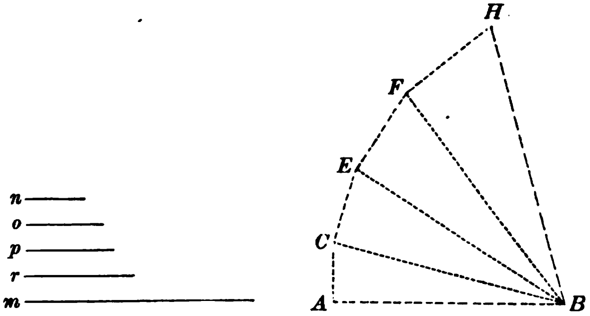
then $S = R' - R$.

$\therefore S$ is the square required.

Q. E. F.

PROPOSITION XVIII. PROBLEM.

348. *To construct a square equivalent to the sum of any number of given squares.*



Let m, n, o, p, r be sides of the given squares.

It is required to construct a square $= m^2 + n^2 + o^2 + p^2 + r^2$.

Take $AB = m$.

Draw $AC = n$ and $\perp$ to AB at A.

Draw BC.

Draw $CE = o$ and $\perp$ to BC at C, and draw BE.

Draw $EF = p$ and $\perp$ to BE at E, and draw BF.

Draw $FH = r$ and $\perp$ to BF at F, and draw BH.

The square constructed on BH is the square required.

For $$\overline{BH}^2 = \overline{FH}^2 + \overline{BF}^2,$$
$$= \overline{FH}^2 + \overline{EF}^2 + \overline{EB}^2,$$
$$= \overline{FH}^2 + \overline{EF}^2 + \overline{EC}^2 + \overline{CB}^2,$$
$$= \overline{FH}^2 + \overline{EF}^2 + \overline{EC}^2 + \overline{CA}^2 + \overline{AB}^2, \text{ § 331}$$

(*the sum of the squares on two sides of a rt.* △ *is equivalent to the square on the hypotenuse*).

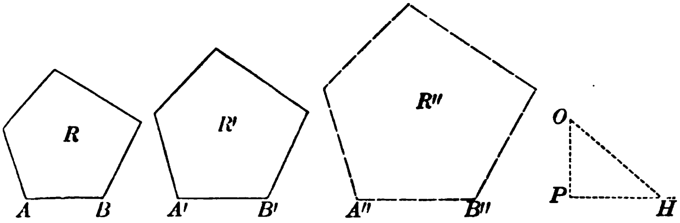
Substitute for AB, CA, EC, EF, and FH, m, n, o, p, and r respectively;

then $$\overline{BH}^2 = m^2 + n^2 + o^2 + p^2 + r^2.$$

Q. E. F.

PROPOSITION XIX. PROBLEM.

349. *To construct a polygon similar to two given similar polygons and equivalent to their sum.*



Let R and R' be two similar polygons, and AB and $A'B'$ two homologous sides.

It is required to construct a similar polygon equivalent to $R + R'$.

Construct the rt. ∠ P.

Take $PH = A'B'$, and $PO = AB$.

Draw OH.

Take $A''B'' = OH$.

Upon $A''B''$, homologous to AB, construct the polygon R'' similar to R.

Then R'' is the polygon required.

For $R' : R :: \overline{A'B'}^2 : \overline{AB}^2$, § 343

(*similar polygons are to each other as the squares on their homologous sides*).

Also $R'' : R' :: \overline{A''B''}^2 : \overline{A'B'}^2$. § 343

In the first proportion, by composition,

$$R' + R : R' :: \overline{A'B'}^2 + \overline{AB}^2 : \overline{A'B'}^2, \quad \text{§ 264}$$

$$:: \overline{PH}^2 + \overline{PO}^2 : \overline{PH}^2,$$

$$:: \overline{HO}^2 : \overline{PH}^2.$$

But $R'' : R' :: \overline{A''B''}^2 : \overline{A'B'}^2,$

$$:: \overline{HO}^2 : \overline{PH}^2.$$

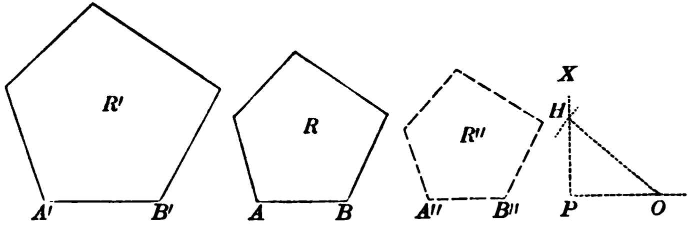
$$\therefore R'' : R' :: R' + R : R';$$

$$\therefore R'' = R' + R.$$

Q. E. F.

PROPOSITION XX. PROBLEM.

350. *To construct a polygon similar to two given similar polygons and equivalent to their difference.*



Let R and R' be two similar polygons, and AB and $A'B'$ two homologous sides.

It is required to construct a similar polygon which shall be equivalent to $R' - R$.

Construct the rt. $\angle P$,

and take $PO = AB$.

From O as a centre, with a radius equal to $A'B'$,

describe an arc cutting PX at H.

Draw OH.

Take $A''B'' = PH$.

On $A''B''$, homologous to AB, construct the polygon R'' similar to R.

Then R'' is the polygon required.

For $\quad R' : R :: \overline{A'B'}^2 : \overline{AB}^2,$ § 343

(*similar polygons are to each other as the squares on their homologous sides*).

Also $\quad R'' : R :: \overline{A''B''}^2 : \overline{AB}^2.$ § 343

In the first proportion, by division,

$$R' - R : R :: \overline{A'B'}^2 - \overline{AB}^2 : \overline{AB}^2, \qquad \text{§ 265}$$
$$:: \overline{OH}^2 - \overline{OP}^2 : \overline{OP}^2,$$
$$:: \overline{PH}^2 : \overline{OP}^2.$$

But $\quad R'' : R :: \overline{A''B''}^2 : \overline{AB}^2,$

$$:: \overline{PH}^2 : \overline{OP}^2.$$

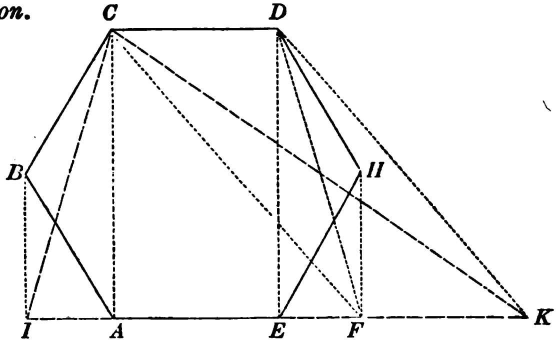
$$\therefore R'' : R :: R' - R : R;$$

$$\therefore R'' = R' - R.$$

Q. E. F.

PROPOSITION XXI. PROBLEM.

351. *To construct a triangle equivalent to a given polygon.*



Let $ABCDHE$ be the given polygon.

It is required to construct a triangle equivalent to the given polygon.

From D draw DE, and from H draw $HF \parallel$ to DE.

Produce AE to meet HF at F, and draw DF.

The polygon $ABCDF$ has one side less than the polygon $ABCDHE$, but the two are equivalent.

For the part $ABCDE$ is common,

and the $\triangle DEF = \triangle DEH$, for the base DE is common, and their vertices F and H are in the line $FH \parallel$ to the base, § 325

(*▲ having the same base and equal altitudes are equivalent*).

Again, draw CF, and draw $DK \parallel$ to CF to meet AF produced at K.

Draw CK.

The polygon $ABCK$ has one side less than the polygon $ABCDF$, but the two are equivalent.

For the part $ABCF$ is common,

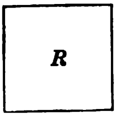
and the $\triangle CFK = \triangle CFD$, for the base CF is common, and their vertices K and D are in the line $KD \parallel$ to the base. § 325

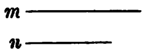
In like manner we may continue to reduce the number of sides of the polygon until we obtain the $\triangle CIK$.

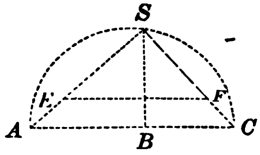
Q. E. F.

PROPOSITION XXII. PROBLEM.

352. *To construct a square which shall have a given ratio to a given square.*







Let R be the given square, and $\frac{n}{m}$ the given ratio.

It is required to construct a square which shall be to R as n is to m.

On a straight line take $AB = m$, and $BC = n$.

On AC as a diameter, describe a semicircle.

At B erect the $\perp$ BS, and draw SA and SC.

Then the $\triangle$ ASC is a rt. $\triangle$ with the rt. $\angle$ at S, § 204

(being inscribed in a semicircle.)

On SA, or SA produced, take SE equal to a side of R.

Draw $EF \parallel$ to AC.

Then SF is a side of the square required.

For $$\frac{\overline{SA}^2}{\overline{SC}^2} = \frac{AB}{BC},$$ § 289

(the squares on the sides of a rt. $\triangle$ have the same ratio as the segments of the hypotenuse made by the $\perp$ let fall from the vertex of the rt. $\angle$).

Also $$\frac{SA}{SC} = \frac{SE}{SF},$$ § 275

(a straight line drawn through two sides of a $\triangle$, parallel to the third side, divides those sides proportionally).

Square the last equality;

then $$\frac{\overline{SA}^2}{\overline{SC}^2} = \frac{\overline{SE}^2}{\overline{SF}^2}.$$

Substitute, in the first equality, for $\frac{\overline{SA}^2}{\overline{SC}^2}$ its equal $\frac{\overline{SE}^2}{\overline{SF}^2}$;

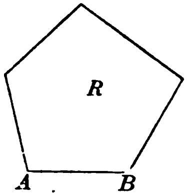
then $$\frac{\overline{SE}^2}{\overline{SF}^2} = \frac{AB}{BC} = \frac{m}{n},$$

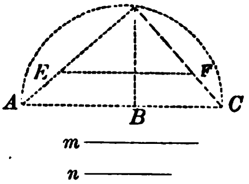
that is, the square having a side equal to SF will have the same ratio to the square R, as n has to m.

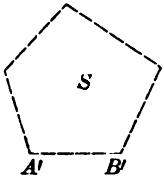
Q. E. F.

PROPOSITION XXIII. PROBLEM.

353. *To construct a polygon similar to a given polygon and having a given ratio to it.*







Let R be the given polygon and $\frac{n}{m}$ the given ratio.

It is required to construct a polygon similar to R, which shall be to R as n is to m.

Find a line, $A'B'$, such that the square constructed upon it shall be to the square constructed upon AB as n is to m. § 352

Upon $A'B'$ as a side homologous to AB, construct the polygon S similar to R.

Then S is the polygon required.

For $$\frac{S}{R} = \frac{\overline{A'B'}^2}{\overline{AB}^2}, \qquad \text{§ 343}$$

(*similar polygons are to each other as the squares on their homologous sides*).

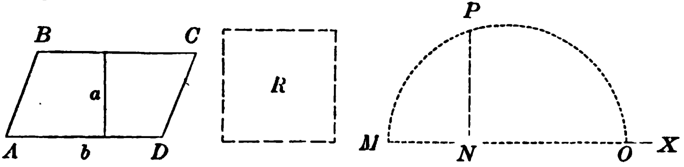
But $$\frac{\overline{A'B'}^2}{\overline{AB}^2} = \frac{n}{m}; \qquad \text{Cons.}$$

$$\therefore \frac{S}{R} = \frac{n}{m}, \quad \text{or,} \quad S : R :: n : m.$$

Q. E. F.

PROPOSITION XXIV. PROBLEM.

354. *To construct a square equivalent to a given parallelogram.*



Let $ABCD$ be a parallelogram, b its base, and a its altitude.

It is required to construct a square $=$ ▱ $ABCD$.

Upon the line MX take $MN = a$, and $NO = b$.

Upon MO as a diameter, describe a semicircle.

At N erect $NP \perp$ to MO.

Then the square R, constructed upon a line equal to NP, is equivalent to the ▱ $ABCD$.

For $MN : NP :: NP : NO$, § 307

(*a* ⊥ *let fall from any point of a circumference to the diameter is a mean proportional between the segments of the diameter*).

$\therefore \overline{NP}^2 = MN \times NO = a \times b$, § 259

(*the product of the means is equal to the product of the extremes*).

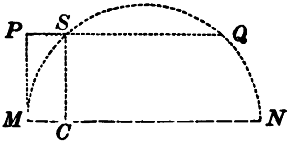
Q. E. F.

355. COROLLARY 1. A square may be constructed equivalent to a triangle, by taking for its side a mean proportional between the base and one-half the altitude of the triangle.

356. COR. 2. A square may be constructed equivalent to any polygon, by first reducing the polygon to an equivalent triangle, and then constructing a square equivalent to the triangle.

PROPOSITION XXV. PROBLEM.

357. *To construct a parallelogram equivalent to a given square, and having the sum of its base and altitude equal to a given line.*



Let R be the given square, and let the sum of the base and altitude of the required parallelogram be equal to the given line MN.

It is required to construct a ▱ $= R$, *and having the sum of its base and altitude* $= MN$.

Upon MN as a diameter, describe a semicircle.

At M erect a $\perp$ MP, equal to a side of the given square R.

Draw PQ $\parallel$ to MN, cutting the circumference at S.

Draw $SC \perp$ to MN.

Any ▱ having CM for its altitude and CN for its base, is equivalent to R.

For SC is $\parallel$ to PM, § 65

(two straight lines $\perp$ to the same straight line are $\parallel$).

$\therefore SC = PM$, § 135

($\parallel$s comprehended between $\parallel$s are equal).

$$\therefore \overline{SC}^2 = \overline{PM}^2 = R.$$

But $MC : SC :: SC : CN$, § 307

(a $\perp$ let fall from any point in a circumference to the diameter is a mean proportional between the segments of the diameter).

Then $\overline{SC}^2 = MC \times CN$, § 259

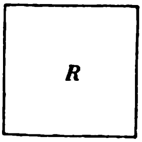
(the product of the means is equal to the product of the extremes).

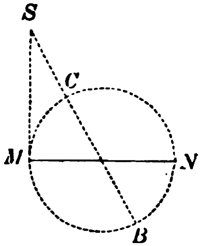
Q. E. F.

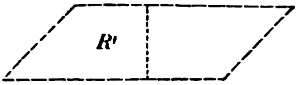
358. SCHOLIUM. The problem is impossible when the side of the square is greater than one-half the line MN.

PROPOSITION XXVI. PROBLEM.

359. *To construct a parallelogram equivalent to a given square, and having the difference of its base and altitude equal to a given line.*







Let R be the given square, and let the difference of the base and altitude of the required parallelogram be equal to the given line MN.

It is required to construct a ▱ = R, with the difference of the base and altitude = MN.

Upon the given line MN as a diameter, describe a circle.

From M draw MS, tangent to the ⊙, and equal to a side of the given square R.

Through the centre of the ⊙, draw SB intersecting the circumference at C and B.

Then any ▱, as R', having SB for its base and SC for its altitude, is equivalent to R.

For $SB : SM :: SM : SC$, § 292

(if from a point without a ⊙, a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and the part without the ⊙).

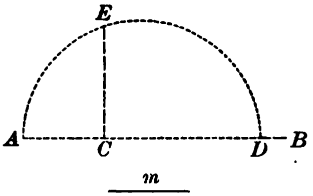
Then $\overline{SM}^2 = SB \times SC$; § 259

and the difference between SB and SC is the diameter of the ⊙, that is, MN.

Q. E. F.

PROPOSITION XXVII. PROBLEM.

360. *Given* $x = \sqrt{2}$, *to construct* x.



Let m represent the unit of length.

It is required to find a line which shall represent the square root of 2.

On the indefinite line AB, take $AC = m$, and $CD = 2\,m$.

On AD as a diameter describe a semi-circumference.

At C erect a ⊥ to AB, intersecting the circumference at E.

Then CE is the line required.

For $AC : CE :: CE : CD$, § 307

(*the ⊥ let fall from any point in the circumference to the diameter, is a mean proportional between the segments of the diameter*);

$$\therefore \overline{CE}^2 = AC \times CD, \qquad \text{§ 259}$$

$$\therefore CE = \sqrt{AC \times CD},$$

$$= \sqrt{1 \times 2} = \sqrt{2}.$$

Q. E. F.

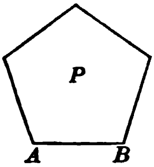
Ex. 1. Given $x = \sqrt{5}$, $y = \sqrt{7}$, $z = 2\sqrt{3}$; to construct x, y, and z.

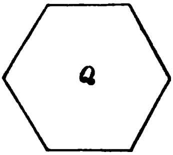
2. Given $2 : x :: x : 3$; to construct x.

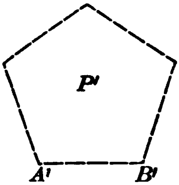
3. Construct a square equivalent to a given hexagon.

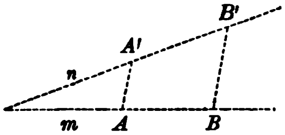
PROPOSITION XXVIII. PROBLEM.

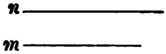
361. *To construct a polygon similar to a given polygon P, and equivalent to a given polygon Q.*











Let *P* and *Q* be two given polygons, and *A B* a side of polygon *P*.

It is required to construct a polygon similar to P and equivalent to Q.

Find a square equivalent to *P*, § 356

and let *m* be equal to one of its sides.

Find a square equivalent to *Q*, § 356

and let *n* be equal to one of its sides.

Find a fourth proportional to *m*, *n*, and *A B*. § 304

Let this fourth proportional be *A' B'*.

Upon *A' B'*, homologous to *A B*, construct the polygon *P'* similar to the given polygon *P*.

Then *P'* is the polygon required.

For $\frac{m}{n} = \frac{AB}{A'B'}$. Cons.

Squaring, $\frac{m^2}{n^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$.

But $P = m^2$, Cons.

and $Q = n^2$; Cons.

$$\therefore \frac{P}{Q} = \frac{m^2}{n^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

But $\frac{P}{P'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$, § 343

(*similar polygons are to each other as the squares on their homologous sides*);

$$\therefore \frac{P}{Q} = \frac{P}{P'};$$ Ax. 1

$\therefore P'$ is equivalent to Q, and is similar to P by construction.

Q. E. F.

Ex. 1. Construct a square equivalent to the sum of three given squares whose sides are respectively 2, 3, and 5.

2. Construct a square equivalent to the difference of two given squares whose sides are respectively 7 and 3.

3. Construct a square equivalent to the sum of a given triangle and a given parallelogram.

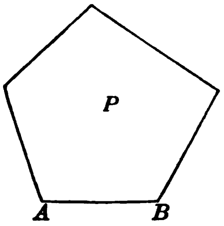
4. Construct a rectangle having the difference of its base and altitude equal to a given line, and its area equivalent to the sum of a given triangle and a given pentagon.

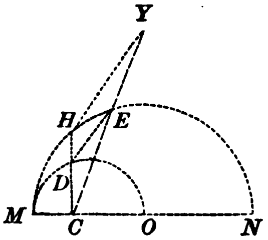
5. Given a hexagon; to construct a similar hexagon whose area shall be to that of the given hexagon as 3 to 2.

6. Construct a pentagon similar to a given pentagon and equivalent to a given trapezoid.

PROPOSITION XXIX. PROBLEM.

362. *To construct a polygon similar to a given polygon, and having two and a half times its area.*





Let P be the given polygon.

It is required to construct a polygon similar to P, and equivalent to $2\frac{1}{2}$ *P.*

Let AB be a side of the given polygon P.

Then $\sqrt{1} : \sqrt{2\frac{1}{2}} :: AB : x$,

or $\sqrt{2} : \sqrt{5} :: AB : x$, § 345

(the homologous sides of similar polygons are to each other as the square roots of their areas).

Take any convenient unit of length, as MC, and apply it six times to the indefinite line MN.

On MO ($= 3\,MC$) describe a semi-circumference;

and on MN ($= 6\,MC$) describe a semi-circumference.

At C erect a ⊥ to MN, intersecting the semi-circumferences at D and H.

Then CD is the $\sqrt{2}$, and CH is the $\sqrt{5}$. § 360

Draw CY, making any convenient ∠ with CH.

On CY take $CE = AB$.

From D draw DE,

and from H draw $HY \parallel$ to DE.

Then CY will equal x, and be a side of the polygon required, homologous to AB.

For $CD : CH :: CE : CY$, § 275

(*a line drawn through two sides of a △, ∥ to the third side, divides the two sides proportionally*).

Substitute their equivalents for CD, CH, and CE;

then $\sqrt{2} : \sqrt{5} :: AB : CY$.

On CY, homologous to AB, construct a polygon similar to the given polygon P;

and this is the polygon required.

Q. E. F.

O Ex. 1. The perpendicular distance between two parallels is 30, and a line is drawn across them at an angle of 45°; what is its length between the parallels?

2. Given an equilateral triangle each of whose sides is 20; find the altitude of the triangle, and its area.

3. Given the angle A of a triangle equal to $\frac{2}{3}$ of a right angle, the angle B equal to $\frac{1}{3}$ of a right angle, and the side a, opposite the angle A, equal to 10; construct the triangle.

4. The two segments of a chord intersected by another chord are 6 and 5, and one segment of the other chord is 3; what is the other segment of the latter chord?

5. If a circle be inscribed in a right triangle: show that the difference between the sum of the two sides containing the right angle and the hypotenuse is equal to the diameter of the circle.

6. Construct a parallelogram the area and perimeter of which shall be respectively equal to the area and perimeter of a given triangle.

7. Given the difference between the diagonal and side of a square; construct the square.

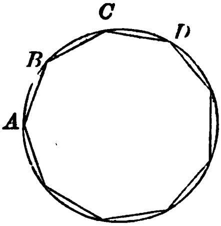
BOOK V.

REGULAR POLYGONS AND CIRCLES.

363. **Def.** A *Regular Polygon* is a polygon which is equilateral and equiangular.

Proposition I. Theorem.

364. ***Every equilateral polygon inscribed in a circle is a regular polygon.***



Let ABC, etc., be an equilateral polygon inscribed in a circle.

We are to prove the polygon ABC, etc., regular.

The arcs AB, BC, CD, etc., are equal, § 182
(*in the same* ⊙, *equal chords subtend equal arcs*).

∴ arcs ABC, BCD, etc., are equal, Ax. 6

∴ the $\measuredangle$s A, B, C, etc., are equal,
(*being inscribed in equal segments*).

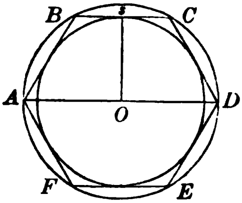
∴ the polygon ABC, etc., is a regular polygon, being equilateral and equiangular.

Q. E. D.

PROPOSITION II. THEOREM.

365. I. *A circle may be circumscribed about a regular polygon.*

II. *A circle may be inscribed in a regular polygon.*



Let $A\,B\,C\,D$, etc., be a regular polygon.

We are to prove that a ⊙ may be circumscribed about this regular polygon, and also a ⊙ may be inscribed in this regular polygon.

CASE I. — Describe a circumference passing through A, B, and C.

From the centre O, draw $O\,A$, $O\,D$,

and draw $O\,s \perp$ to chord $B\,C$.

On $O\,s$ as an axis revolve the quadrilateral $O\,A\,B\,s$,

until it comes into the plane of $O\,s\,C\,D$.

The line $s\,B$ will fall upon $s\,C$,

(for $\angle\, O\,s\,B = \angle\, O\,s\,C$, both being rt. ∡).

The point B will fall upon C, § 183

(since $s\,B = s\,C$).

The line $B\,A$ will fall upon $C\,D$, § 363

(since $\angle\, B = \angle\, C$, being ∡ of a regular polygon).

The point A will fall upon D, § 363

(since $B\,A = C\,D$, being sides of a regular polygon).

∴ the line $O\,A$ will coincide with line $O\,D$,

(their extremities being the same points).

∴ the circumference will pass through D.

In like manner we may prove that the circumference, passing through vertices B, C, and D will also pass through the vertex E, and thus through all the vertices of the polygon in succession.

CASE II. — The sides of the regular polygon, being equal chords of the circumscribed ⊙, are equally distant from the centre, § 185

∴ a circle described with the centre O and a radius $O\,s$ will touch all the sides, and be inscribed in the polygon. § 174

Q. E. D.

366. Def. The *Centre* of a regular polygon is the common centre O of the circumscribed and inscribed circles.

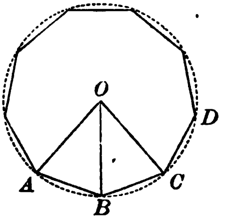
367. Def. The *Radius* of a regular polygon is the radius $O A$ of the circumscribed circle.

368. Def. The *Apothem* of a regular polygon is the radius $O s$ of the inscribed circle.

369. Def. The *Angle at the centre* is the angle included by the radii drawn to the extremities of any side.

Proposition III. Theorem.

370. ***Each angle at the centre of a regular polygon is equal to four right angles divided by the number of sides of the polygon.***



Let $A B C$, etc., be a regular polygon of n sides.

We are to prove $\angle A O B = \frac{4 \text{ rt. } \angle s}{n}$.

Circumscribe a ⊙ about the polygon.

The $\angle s$ $A O B$, $B O C$, etc., are equal, § 180
(*in the same ⊙ equal arcs subtend equal* $\angle s$ *at the centre*).

$\therefore$ the $\angle A O B = 4$ rt. $\angle s$ divided by the number of $\angle s$ about O.

But the number of $\angle s$ about $O = n$, the number of sides of the polygon.

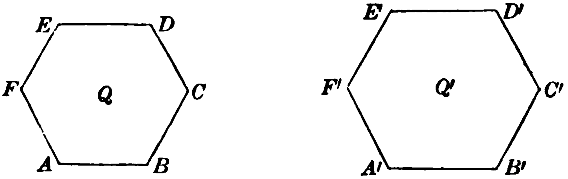
$$\therefore \angle A O B = \frac{4 \text{ rt. } \angle s}{n}.$$

Q. E. D.

371. Corollary. The radius drawn to any vertex of a regular polygon bisects the angle at that vertex.

PROPOSITION IV. THEOREM.

372. *Two regular polygons of the same number of sides are similar.*



Let Q and Q' be two regular polygons, each having n sides.

We are to prove Q *and* Q' *similar polygons.*

The sum of the interior $\angle$s of each polygon is equal to 2 rt. $\angle$s $(n-2)$, § 157

(*the sum of the interior $\angle$s of a polygon is equal to 2 rt. $\angle$s taken as many times less 2 as the polygon has sides*).

$$\text{Each } \angle \text{ of the polygon } Q = \frac{2 \text{ rt. } \angle\text{s } (n-2)}{n}, \qquad \S\ 158$$

(*for the $\angle$s of a regular polygon are all equal, and hence each $\angle$ is equal to the sum of the $\angle$s divided by their number*).

$$\text{Also, each } \angle \text{ of } Q' = \frac{2 \text{ rt. } \angle\text{s } (n-2)}{n}. \qquad \S\ 158$$

$\therefore$ the two polygons Q and Q' are mutually equiangular.

Moreover, $$\frac{AB}{BC} = 1, \qquad \S\ 363$$

(*the sides of a regular polygon are all equal*);

and $$\frac{A'B'}{B'C'} = 1, \qquad \S\ 363$$

$$\therefore \frac{AB}{BC} = \frac{A'B'}{B'C'}, \qquad \text{Ax. 1}$$

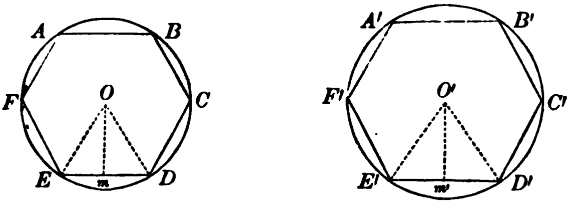
$\therefore$ the two polygons have their homologous sides proportional;

$\therefore$ the two polygons are similar. § 278

Q. E. D.

PROPOSITION V. THEOREM.

373. *The homologous sides of similar regular polygons have the same ratio as the radii of their circumscribed circles, and, also as the radii of their inscribed circles.*



Let O and O' be the centres of the two similar regular polygons ABC, etc., and $A'B'C'$, etc.

From O and O' draw OE, OD, $O'E'$, $O'D'$, also the ⊥s Om and $O'm'$.

OE and $O'E'$ are radii of the circumscribed ⓢ, § 367

and Om and $O'm'$ are radii of the inscribed ⓢ. § 368

We are to prove $\frac{ED}{E'D'} = \frac{OE}{O'E'} = \frac{Om}{O'm'}$.

In the ▵s OED and $O'E'D'$

the ∠s OED, ODE, $O'E'D'$ and $O'D'E'$ are equal, § 371

(*being halves of the equal ∠s FED, EDC, $F'E'D'$ and $E'D'C'$*);

∴ the ▵s OED and $O'E'D'$ are similar, § 280

(*if two ▵s have two ∠s of the one equal respectively to two ∠s of the other, they are similar*).

$$\therefore \frac{ED}{E'D'} = \frac{OE}{O'E'}, \qquad \text{§ 278}$$

(*the homologous sides of similar ▵s are proportional*).

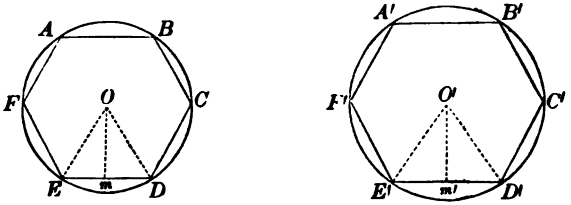
Also, $$\frac{ED}{E'D'} = \frac{Om}{O'm'}, \qquad \text{§ 297}$$

(*the homologous altitudes of similar ▵s have the same ratio as their homologous bases*).

Q. E. D.

PROPOSITION VI. THEOREM.

374. *The perimeters of similar regular polygons have the same ratio as the radii of their circumscribed circles, and, also as the radii of their inscribed circles.*



Let P and P' represent the perimeters of the two similar regular polygons ABC, etc., and $A'B'C'$, etc.

From centres O, O' draw OE, $O'E'$, and ⊥s Om and $O'm'$.

We are to prove $\frac{P}{P'} = \frac{OE}{O'E'} = \frac{Om}{O'm'}$.

$$\frac{P}{P'} = \frac{ED}{E'D'}, \qquad \S\ 295$$

(*the perimeters of similar polygons have the same ratio as any two homologous sides*).

Moreover, $$\frac{OE}{O'E'} = \frac{ED}{E'D'}, \qquad \S\ 373$$

(*the homologous sides of similar regular polygons have the same ratio as the radii of their circumscribed* Ⓢ).

Also $$\frac{Om}{O'm'} = \frac{ED}{E'D'}, \qquad \S\ 373$$

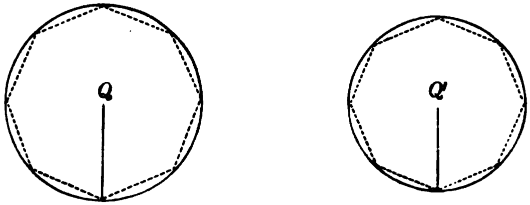
(*the homologous sides of similar regular polygons have the same ratio as the radii of their inscribed* Ⓢ).

$$\therefore \frac{P}{P'} = \frac{OE}{O'E'} = \frac{Om}{O'm'}.$$

Q. E. D

Proposition VII. Theorem.

375. *The circumferences of circles have the same ratio as their radii.*



Let C and C' be the circumferences, R and R' the radii of the two circles Q and Q'.

We are to prove $C : C' :: R : R'$.

Inscribe in the Ⓢ two regular polygons of the same number of sides.

Conceive the number of the sides of these similar regular polygons to be indefinitely increased, the polygons continuing to be inscribed, and to have the same number of sides.

Then the perimeters will continue to have the same ratio as the radii of their circumscribed circles, § 374

(*the perimeters of similar regular polygons have the same ratio as the radii of their circumscribed* Ⓢ),

and will approach indefinitely to the circumferences as their limits.

∴ the circumferences will have the same ratio as the radii of their circles, § 199

$$\therefore C : C' :: R : R'.$$

Q. E. D.

376. Corollary. By multiplying by 2, both terms of the ratio $R : R'$, we have

$$C : C' :: 2R : 2R';$$

that is, the circumferences of circles are to each other as their diameters.

Since $C : C' :: 2R : 2R'$,

$C : 2R :: C' : 2R'$, § 262

or, $\dfrac{C}{2R} = \dfrac{C'}{2R'}$.

That is, the ratio of the circumference of a circle to its diameter is a constant quantity.

This constant quantity is denoted by the Greek letter π.

377. Scholium. The ratio π is incommensurable, and therefore can be expressed only approximately in figures. The letter π, however, is used to *represent* its *exact* value.

Ex. 1. Show that two triangles which have an angle of the one equal to the supplement of the angle of the other are to each other as the products of the sides including the supplementary angles.

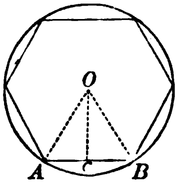
2. Show, geometrically, that the square described upon the sum of two straight lines is equivalent to the sum of the squares described upon the two lines *plus* twice their rectangle.

3. Show, geometrically, that the square described upon the difference of two straight lines is equivalent to the sum of the squares described upon the two lines *minus* twice their rectangle.

4. Show, geometrically, that the rectangle of the sum and difference of two straight lines is equivalent to the difference of the squares on those lines.

PROPOSITION VIII. THEOREM.

378. *If the number of sides of a regular inscribed polygon be increased indefinitely, the apothem will be an increasing variable whose limit is the radius of the circle.*



In the right triangle OCA, let OA be denoted by R, OC by r, and AC by b.

We are to prove lim. $(r) = R$.

$r < R$, § 52

(*a* ⊥ *is the shortest distance from a point to a straight line*).

And $R - r < b$, § 97

(*one side of a* △ *is greater than the difference of the other two sides*).

By increasing the number of sides of the polygon indefinitely, AB, that is, $2b$, can be made less than any assigned quantity.

∴ b, *the half* of $2b$, can be made less than any assigned quantity.

∴ $R - r$, which is *less* than b, can be made less than any assigned quantity.

$$\therefore \textit{lim.}\ (R - r) = 0.$$

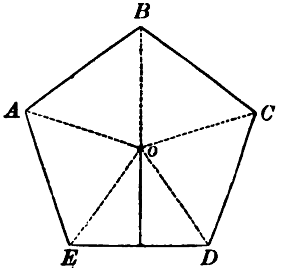
$$\therefore R - \textit{lim.}\ (r) = 0. \qquad \S\ 199$$

$$\therefore \textit{lim.}\ (r) = R.$$

Q. E. D.

PROPOSITION IX. THEOREM.

379. *The area of a regular polygon is equal to one-half the product of its apothem by its perimeter.*



Let P represent the perimeter and R the apothem of the regular polygon $A B C$, etc.

We are to prove *the area of $A B C$, etc., $= \frac{1}{2} R \times P$.*

Draw $O A$, $O B$, $O C$, etc.

The polygon is divided into as many ▲ as it has sides.

The apothem is the common altitude of these ▲,

and the area of each △ is equal to $\frac{1}{2} R$ multiplied by the base. § 324

∴ the area of all the ▲ is equal to $\frac{1}{2} R$ multiplied by the sum of all the bases.

But the sum of the areas of all the ▲ is equal to the area of the polygon,

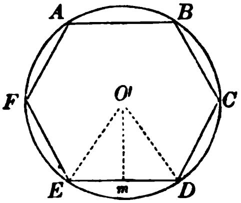
and the sum of all the bases of the ▲ is equal to the perimeter of the polygon.

∴ the area of the polygon $= \frac{1}{2} R \times P$.

Q. E. D.

PROPOSITION X. THEOREM.

380. *The area of a circle is equal to one-half the product of its radius by its circumference.*



Let R represent the radius, and C the circumference of a circle.

We are to prove *the area of the circle* $= \frac{1}{2} R \times C$.

Inscribe any regular polygon, and denote its perimeter by P, and its apothem by r.

Then the area of this polygon $= \frac{1}{2} r \times P$, § 379

(*the area of a regular polygon is equal to one-half the product of its apothem by the perimeter*).

Conceive the number of sides of this polygon to be indefinitely increased, the polygon still continuing to be regular and inscribed.

Then the perimeter of the polygon approaches the circumference of the circle as its limit,

the apothem, the radius as its limit, § 378

and the area of the polygon approaches the ⊙ as its limit.

But the area of the polygon continues to be equal to one-half the product of the apothem by the perimeter, however great the number of sides of the polygon.

∴ the area of the ⊙ $= \frac{1}{2} R \times C$. § 199

Q. E. D.

381. Corollary 1. Since $\frac{C}{2R} = \pi$, § 376

$$\therefore C = 2\pi R.$$

In the equality, the area of the $\odot = \frac{1}{2} R \times C$,

substitute $2\pi R$ for C;

then the area of the $\odot = \frac{1}{2} R \times 2\pi R$,
$= \pi R^2$.

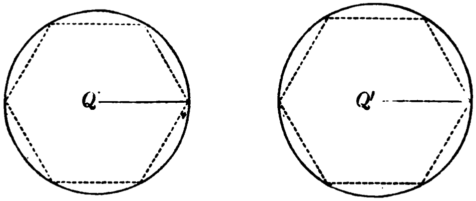
That is, *the area of a* $\odot = \pi$ *times the square on its radius.*

382. Cor. 2. *The area of a sector equals* $\frac{1}{2}$ *the product of its radius by its arc;* for the sector is such part of the circle as its arc is of the circumference.

383. Def. In different circles *similar arcs, similar sectors,* and *similar segments,* are such as correspond to equal angles at the centre.

Proposition XI. Theorem.

384. *Two circles are to each other as the squares on their radii.*



Let R and R' be the radii of the two circles Q and Q'.

We are to prove $\frac{Q}{Q'} = \frac{R^2}{R'^2}$.

Now $Q = \pi R^2$, § 381

(*the area of a* $\odot = \pi$ *times the square on its radius*),

and $Q' = \pi R'^2$. § 381

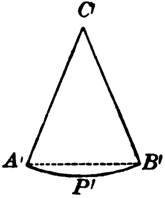
Then $\frac{Q}{Q'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2}$.

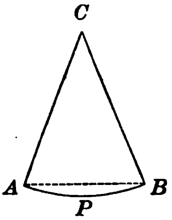
Q. E. D.

385. Corollary. *Similar arcs,* being like parts of their respective circumferences, *are to each other as their radii; similar sectors,* being like parts of their respective circles, *are to each other as the squares on their radii.*

PROPOSITION XII. THEOREM.

386. *Similar segments are to each other as the squares on their radii.*





Let AC and $A'C'$ be the radii of the two similar segments ABP and $A'B'P'$.

We are to prove $\dfrac{ABP}{A'B'P'} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$.

The sectors ACB and $A'C'B'$ are similar, § 383
(having the ∠s at the centre, C and C', equal).

In the $\triangle$s ACB and $A'C'B'$

$\angle C = \angle C'$, § 383
(being corresponding ∠s of similar sectors).

$AC = CB$, § 163

$A'C' = C'B'$; § 163

∴ the $\triangle$s ACB and $A'C'B'$ are similar, § 284
(having an ∠ of the one equal to an ∠ of the other, and the including sides proportional).

Now $\dfrac{\text{sector } ACB}{\text{sector } A'C'B'} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$, § 385
(similar sectors are to each other as the squares on their radii);

and $\dfrac{\triangle ACB}{\triangle A'C'B'} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$, § 342
(similar $\triangle$s are to each other as the squares on their homologous sides).

Hence $\dfrac{\text{sector } ACB - \triangle ACB}{\text{sector } A'C'B' - \triangle A'C'B'} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$,

or, $\dfrac{\text{segment } ABP}{\text{segment } A'B'P'} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$, § 271
(if two quantities be increased or diminished by like parts of each, the results will be in the same ratio as the quantities themselves).

Q. E. D.

EXERCISES.

1. Show that an equilateral polygon circumscribed about a circle is regular if the number of its sides be *odd*.

2. Show that an equiangular polygon inscribed in a circle is regular if the number of its sides be *odd*.

3. Show that *any* equiangular polygon circumscribed about a circle is regular.

4. Show that the side of a circumscribed equilateral triangle is double the side of an inscribed equilateral triangle.

5. Show that the area of a regular inscribed hexagon is three-fourths of that of the regular circumscribed hexagon.

6. Show that the area of a regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

7. Show that the area of a regular inscribed octagon is equal to that of a rectangle whose adjacent sides are equal to the sides of the inscribed and circumscribed squares.

8. Show that the area of a regular inscribed dodecagon is equal to three times the square on the radius.

9. Given the diameter of a circle 50; find the area of the circle. Also, find the area of a sector of 80° of this circle.

10. Three equal circles touch each other externally and thus inclose one acre of ground; find the radius in rods of each of these circles.

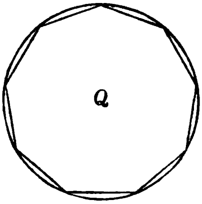
11. Show that in two circles of different radii, angles at the centres subtended by arcs of equal length are to each other inversely as the radii.

12. Show that the square on the side of a regular inscribed pentagon, minus the square on the side of a regular inscribed decagon, is equal to the square on the radius.

ON CONSTRUCTIONS.

PROPOSITION XIII. PROBLEM.

387. *To inscribe a regular polygon of any number of sides in a given circle.*



Let Q be the given circle, and n the number of sides of the polygon.

It is required to inscribe in Q, a regular polygon having n sides.

Divide the circumference of the ⊙ into n equal arcs.

Join the extremities of these arcs.

Then we have the polygon required.

For the polygon is equilateral, § 181

(*in the same ⊙ equal arcs are subtended by equal chords*);

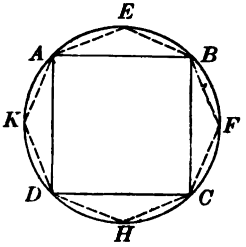
and the polygon is also regular, § 364

(*an equilateral polygon inscribed in a ⊙ is regular*).

Q. E. F.

Proposition XIV. Problem.

388. *To inscribe in a given circle a regular polygon which has double the number of sides of a given inscribed regular polygon.*



Let $A B C D$ be the given inscribed polygon.

It is required to inscribe a regular polygon having double the number of sides of $A B C D$.

Bisect the arcs $A B$, $B C$, etc.

Draw $A E$, $E B$, $B F$, etc.,

The polygon $A E B F C$, etc., is the polygon required.

For the chords $A B$, $B C$, etc., are equal, § 363

(being sides of a regular polygon).

∴ the arcs $A B$, $B C$, etc., are equal, § 182

(in the same ⊙ equal chords subtend equal arcs).

Hence the halves of these arcs are equal,

or, $A E$, $E B$, $B F$, $F C$, etc., are equal;

∴ the polygon $A E B F$, etc., is equilateral.

The polygon is also regular, § 364

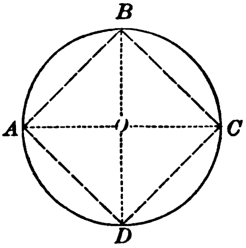
(an equilateral polygon inscribed in a ⊙ is regular);

and has double the number of sides of the given regular polygon.

Q. E. F.

PROPOSITION XV. PROBLEM.

389. *To inscribe a square in a given circle.*



Let O be the centre of the given circle.

It is required to inscribe a square in the circle.

Draw the two diameters AC and BD ⊥ to each other.

Join AB, BC, CD, and DA.

Then $ABCD$ is the square required.

For, the ⦣ ABC, BCD, etc., are rt. ⦣, § 204
(being inscribed in a semicircle),

and the sides AB, BC, etc., are equal, § 181
(in the same ⊙ equal arcs are subtended by equal chords);

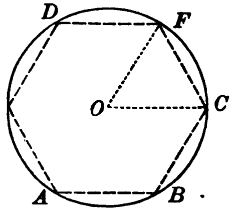
∴ the figure $ABCD$ is a square, § 127
(having its sides equal and its ⦣ rt. ⦣).

Q. E. F.

390. COROLLARY. By bisecting the arcs AB, BC, etc., a regular polygon of 8 sides may be inscribed; and, by continuing the process, regular polygons of 16, 32, 64, etc., sides may be inscribed.

PROPOSITION XVI. PROBLEM.

391. *To inscribe in a given circle a regular hexagon.*



Let O be the centre of the given circle.

It is required to inscribe in the given ⊙ a regular hexagon.

From O draw any radius, as $O\,C$.

From C as a centre, with a radius equal to $O\,C$,

describe an arc intersecting the circumference at F.

Draw $O\,F$ and $C\,F$.

Then $C\,F$ is a side of the regular hexagon required.

For the △ $O\,F\,C$ is equilateral, Cons.

and equiangular, § 112

∴ the ∠ $F\,O\,C$ is $\frac{1}{3}$ of 2 rt. ∠s, or, $\frac{1}{6}$ of 4 rt. ∠s. § 98

∴ the arc $F\,C$ is $\frac{1}{6}$ of the circumference $A\,B\,C\,F$,

∴ the chord $F\,C$, which subtends the arc $F\,C$, is a side of a regular hexagon;

and the figure $C\,F\,D$, etc., formed by applying the radius six times as a chord, is the hexagon required.

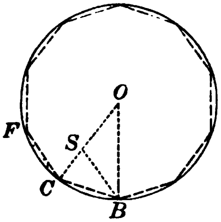
Q. E. F.

392. COROLLARY 1. By joining the alternate vertices A, C, D, an equilateral △ is inscribed in a circle.

393. COR. 2. By bisecting the arcs $A\,B$, $B\,C$, etc., a regular polygon of 12 sides may be inscribed in a circle; and, by continuing the process, regular polygons of 24, 48, etc., sides may be inscribed.

PROPOSITION XVII. PROBLEM.

394. *To inscribe in a given circle a regular decagon.*



Let O be the centre of the given circle.

It is required to inscribe in the given ⊙ *a regular decagon.*

Draw the radius OC,

and divide it in extreme and mean ratio, so that OC shall be to OS as OS is to SC. § 311

From C as a centre, with a radius equal to OS,

describe an arc intersecting the circumference at B.

Draw BC, BS, and BO.

Then BC is a side of the regular decagon required.

For $OC : OS :: OS : SC$, Cons.

and $BC = OS$. Cons.

Substitute for OS its equal BC,

then $OC : BC :: BC : SC$.

Moreover the $\angle\, OCB = \angle\, SCB$, Iden.

∴ the $\triangle$ OCB and BCS are similar, § 284

(having an $\angle$ of the one equal to an $\angle$ of the other, and the including sides proportional).

But the $\triangle$ OCB is isosceles, § 160

(its sides $O\,C$ and $O\,B$ being radii of the same circle).

∴ the $\triangle$ BCS, which is similar to the $\triangle$ OCB, is isosceles, and $BS = BC$. § 114

But $OS = BC$, Cons.

$\therefore OS = BS$, Ax. 1

$\therefore$ the $\triangle SOB$ is isosceles,

and the $\angle O = \angle SBO$, § 112

(being opposite equal sides).

But the $\angle CSB = \angle O + \angle SBO$, § 105

(the exterior $\angle$ of a $\triangle$ is equal to the sum of the two opposite interior $\angle$s).

$\therefore$ the $\angle CSB = 2 \angle O$.

$\angle SCB \, (= \angle CSB) = 2 \angle O$, § 112

and $\angle OBC \, (= \angle SCB) = 2 \angle O$. § 112

$\therefore$ the sum of the $\angle$s of the $\triangle OCB = 5 \angle O$.

$\therefore 5 \angle O = 2$ rt. $\angle$s, § 98

and $\angle O = \frac{1}{5}$ of 2 rt. $\angle$s, or $\frac{1}{10}$ of 4 rt. $\angle$s.

$\therefore$ the arc BC is $\frac{1}{10}$ of the circumference, and

$\therefore$ the chord BC is a side of a regular inscribed decagon.

Hence, to inscribe a regular decagon, divide the radius in extreme and mean ratio, and apply the greater segment ten times as a chord.

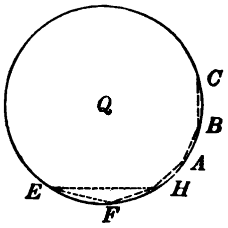
Q. E. F.

395. Corollary 1. By joining the alternate vertices of a regular inscribed decagon, a regular pentagon may be inscribed.

396. Cor. 2. By bisecting the arcs BC, CF, etc., a regular polygon of 20 sides may be inscribed, and, by continuing the process, regular polygons of 40, 80, etc., sides may be inscribed.

PROPOSITION XVIII. PROBLEM.

397. *To inscribe in a given circle a regular pentedecagon, or polygon of fifteen sides.*



Let Q be the given circle.

It is required to inscribe in Q a regular pentedecagon.

Draw EH equal to a side of a regular inscribed hexagon, § 391

and EF equal to a side of a regular inscribed decagon. § 394

Join FH.

Then FH will be a side of a regular inscribed pentedecagon.

For the arc EH is $\frac{1}{6}$ of the circumference,

and the arc EF is $\frac{1}{10}$ of the circumference;

∴ the arc FH is $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$, of the circumference.

∴ the chord FH is a side of a regular inscribed pentedecagon,

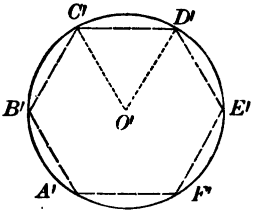
and by applying FH fifteen times as a chord, we have the polygon required.

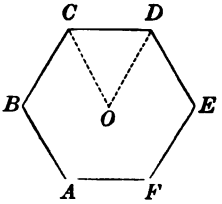
Q. E. F.

398. COROLLARY. By bisecting the arcs FH, HA, etc., a regular polygon of 30 sides may be inscribed; and by continuing the process, regular polygons of 60, 120, etc. sides may be inscribed.

PROPOSITION XIX. PROBLEM.

399. *To inscribe in a given circle a regular polygon similar to a given regular polygon.*





Let $ABCD$, etc., be the given regular polygon, and $C'D'E'$ the given circle.

It is required to inscribe in $C'D'E'$ a regular polygon similar to $ABCD$, etc.

From O, the centre of the polygon $ABCD$, etc.

draw OD and OC.

From O' the centre of the $\odot\, C'D'E'$,

draw $O'C'$ and $O'D'$,

making the $\angle O' = \angle O$.

Draw $C'D'$.

Then $C'D'$ will be a side of the regular polygon required.

For each polygon will have as many sides as the $\angle O$ ($= \angle O'$) is contained times in 4 rt. $\angle$s.

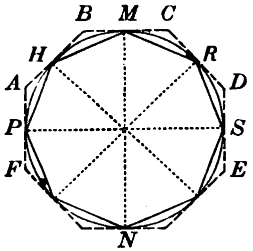
$\therefore$ the polygon $C'D'E'$, etc. is similar to the polygon CDE, etc., § 372

(two regular polygons of the same number of sides are similar).

Q. E. F.

PROPOSITION XX. PROBLEM.

400. *To circumscribe about a circle a regular polygon similar to a given inscribed regular polygon.*



Let HMRS, etc., be a given inscribed regular polygon.

It is required to circumscribe a regular polygon similar to HMRS, etc.

At the vertices H, M, R, etc., draw tangents to the ⊙, intersecting each other at A, B, C, etc.

Then the polygon $ABCD$, etc. will be the regular polygon required.

Since the polygon $ABCD$, etc.

has the same number of sides as the polygon $HMRS$, etc.,

it is only necessary to prove that $ABCD$, etc. is a regular polygon. § 372

In the $\triangle$s BHM and CMR,

$$HM = MR, \qquad \text{§ 363}$$

(being sides of a regular polygon),

the $\angle$s BHM, BMH, CMR, and CRM are equal, § 209

(being measured by halves of equal arcs);

$\therefore$ the $\triangle$s BHM and CMR are equal, § 107

(having a side and two adjacent $\angle$s of the one equal respectively to a side and two adjacent $\angle$s of the other).

$\therefore \angle B = \angle C$,

(being homologous $\angle$s of equal $\triangle$s).

In like manner we may prove $\angle C = \angle D$, etc.

$\therefore$ the polygon $ABCD$, etc., is equiangular.

Since the $\triangle$s BHM, CMR, etc. are isosceles, § 241

(two tangents drawn from the same point to a ⊙ are equal),

the sides BH, BM, CM, CR, etc. are equal,

(being homologous sides of equal isosceles $\triangle$s).

$\therefore$ the sides AB, BC, CD, etc. are equal, Ax. 6

and the polygon $ABCD$, etc. is equilateral.

Therefore the circumscribed polygon is regular and similar to the given inscribed polygon. § 372

Q. E. F.

Ex. Let R denote the radius of a regular inscribed polygon, r the apothem, a one side, A one angle, and C the angle at the centre; show that

1. In a regular inscribed triangle $a = R\sqrt{3}$, $r = \frac{1}{2}R$, $A = 60°$, $C = 120°$.

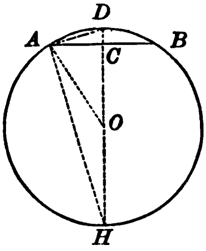
2. In an inscribed square $a = R\sqrt{2}$, $r = \frac{1}{2}R\sqrt{2}$, $A = 90°$, $C = 90°$.

3. In a regular inscribed hexagon $a = R$, $r = \frac{1}{2}R\sqrt{3}$, $A = 120°$, $C = 60°$.

4. In a regular inscribed decagon $a = \frac{R(\sqrt{5} - 1)}{2}$, $r = \frac{1}{4}R\sqrt{10 + 2\sqrt{5}}$, $A = 144°$, $C = 36°$.

Proposition XXI. Problem.

401. *To find the value of the chord of one-half an arc, in terms of the chord of the whole arc and the radius of the circle.*



Let AB be the chord of arc AB and AD the chord of one-half the arc AB.

It is required to find the value of AD in terms of AB and R (radius).

From D draw DH through the centre O,

and draw OA.

HD is ⊥ to the chord AB at its middle point C, § 60

(*two points, O and D, equally distant from the extremities, A and B, determine the position of a ⊥ to the middle point of AB*).

The $\angle HAD$ is a rt. $\angle$, § 204

(*being inscribed in a semicircle*),

$\therefore \overline{AD}^2 = DH \times DC,$ § 289

(*the square on one side of a rt. △ is equal to the product of the hypotenuse by the adjacent segment made by the ⊥ let fall from the vertex of the rt. ∠*).

Now $DH = 2R,$

and $DC = DO - CO = R - CO;$

$$\therefore \overline{AD}^2 = 2R(R - CO).$$

Since ACO is a rt. $\triangle$,

$$\overline{AO}^2 = \overline{AC}^2 + \overline{CO}^2; \qquad \S\ 331$$

$$\therefore \overline{CO}^2 = \overline{AO}^2 - \overline{AC}^2.$$

$$\therefore CO = \sqrt{(\overline{AO}^2 - \overline{AC}^2)},$$

$$= \sqrt{R^2 - (\tfrac{1}{2} AB)^2},$$

$$= \sqrt{R^2 - \tfrac{1}{4}\overline{AB}^2},$$

$$= \sqrt{\frac{4R^2 - \overline{AB}^2}{4}},$$

$$= \frac{\sqrt{4R^2 - \overline{AB}^2}}{2}.$$

In the equation $\overline{AD}^2 = 2R(R - CO)$,

substitute for CO its value $\dfrac{\sqrt{4R^2 - \overline{AB}^2}}{2}$;

then $$\overline{AD}^2 = 2R\left(R - \frac{\sqrt{4R^2 - \overline{AB}^2}}{2}\right),$$

$$= 2R^2 - R\left(\sqrt{4R^2 - \overline{AB}^2}\right).$$

$$\therefore AD = \sqrt{2R^2 - R\left(\sqrt{4R^2 - \overline{AB}^2}\right)}.$$

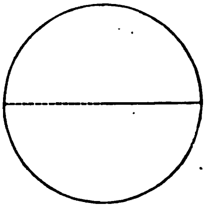
Q. E. F.

402. Corollary. If we take the radius equal to unity, the equation $AD = \sqrt{2R^2 - R\left(\sqrt{4R^2 - \overline{AB}^2}\right)}$ becomes

$$AD = \sqrt{2 - \sqrt{4 - \overline{AB}^2}}.$$

Proposition XXII. Problem.

403. *To compute the ratio of the circumference of a circle to its diameter, approximately.*



Let C be the circumference and R the radius of a circle.

Since $\pi = \dfrac{C}{2R}$, § 376

$$\text{when } R = 1,\ \pi = \frac{C}{2}.$$

It is required to find the numerical value of π.

We make the following computations by the use of the formula obtained in the last proposition,

$$AD = \sqrt{2 - \sqrt{4 - \overline{AB}^2}},$$

when AB is a side of a regular hexagon:

In a polygon of

No. Sides.	Form of Computation.	Length of Side.	Perimeter.
12	$AD = \sqrt{2 - \sqrt{4 - 1^2}}$	.51763809	6.21165708
24	$AD = \sqrt{2 - \sqrt{4 - (.51763809)^2}}$	.26105238	6.26525722
48	$AD = \sqrt{2 - \sqrt{4 - (.26105238)^2}}$	.13080626	6.27870041
96	$AD = \sqrt{2 - \sqrt{4 - (.13080626)^2}}$	.06543817	6.28206396
192	$AD = \sqrt{2 - \sqrt{4 - (.06543817)^2}}$	.03272346	6.28290510
384	$AD = \sqrt{2 - \sqrt{4 - (.03272346)^2}}$	.01636228	6.28311544
768	$AD = \sqrt{2 - \sqrt{4 - (.01636228)^2}}$	.00818121	6.28316941

Hence we may consider 6.28317 as approximately the circumference of a ⊙ whose radius is unity.

$$\therefore \pi, \text{ which equals } \frac{C}{2}, = \frac{6.28317}{2}.$$

$$\therefore \pi = 3.14159 \text{ nearly.}$$

Q. E. F

ON ISOPERIMETRICAL POLYGONS. — SUPPLEMENTARY.

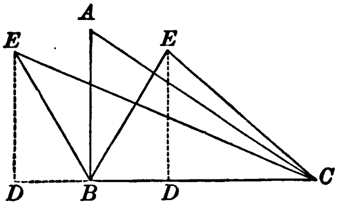
404. DEF. *Isoperimetrical* figures are figures which have equal perimeters.

405. DEF. Among magnitudes of the same kind, that which is greatest is a *Maximum*, and that which is smallest is a *Minimum*.

Thus the diameter of a circle is the maximum among all inscribed straight lines; and a perpendicular is the minimum among all straight lines drawn from a point to a given straight line.

PROPOSITION XXIII. THEOREM.

406. *Of all triangles having two sides respectively equal, that in which these sides include a right angle is the maximum.*



Let the triangles ABC and EBC have the sides AB and BC equal respectively to EB and BC; and let the angle ABC be a right angle.

We are to prove $\triangle ABC > \triangle EBC$.

From E, let fall the $\perp$ ED.

The $\triangle$s ABC and EBC, having the same base BC, are to each other as their altitudes AB and ED, § 326

(*$\triangle$s having the same base are to each other as their altitudes*).

Now ED is $< EB$, § 52

(*a $\perp$ is the shortest distance from a point to a straight line*).

But $EB = AB$, Hyp.

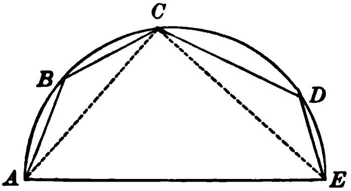
$\therefore ED$ is $< AB$.

$\therefore \triangle ABC > \triangle EBC$.

Q. E. D.

PROPOSITION XXIV. THEOREM.

407. *Of all polygons formed of sides all given but one, the polygon inscribed in a semicircle, having the undetermined side for its diameter, is the maximum.*



Let AB, BC, CD, and DE be the sides of a polygon inscribed in a semicircle having AE for its diameter.

We are to prove the polygon $ABCDE$ the maximum of polygons having the sides AB, BC, CD, and DE.

From any vertex, as C, draw CA and CE.

Then the ∠ ACE is a rt. ∠, § 204

(being inscribed in a semicircle).

Now the polygon is divided into three parts, ABC, CDE, and ACE.

The parts ABC and CDE will remain the same, if the ∠ ACE be increased or diminished;

but the part ACE will be diminished, § 406

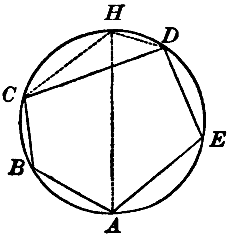
(of all △ having two sides respectively equal, that in which these sides include a rt. ∠ is the maximum).

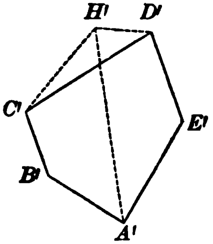
∴ $ABCDE$ is the maximum polygon.

Q. E. D.

Proposition XXV. Theorem.

408. *The maximum of all polygons formed of given sides can be inscribed in a circle.*





Let $ABCDE$ be a polygon inscribed in a circle, and $A'B'C'D'E'$ be a polygon, equilateral with respect to $ABCDE$, but which cannot be inscribed in a circle.

We are to prove

the polygon $ABCDE >$ *the polygon* $A'B'C'D'E'$.

Draw the diameter AH.

Join CH and DH.

Upon $C'D'$ ($= CD$) construct the $\triangle C'H'D' = \triangle CHD$,

and draw $A'H'$.

Now the polygon $ABCH >$ the polygon $A'B'C'H'$, § 407

(of all polygons formed of sides all given but one, the polygon inscribed in a semicircle having the undetermined side for its diameter, is the maximum).

And the polygon $AEDH >$ the polygon $A'E'D'H'$. § 407

Add these two inequalities, then

the polygon $ABCHDE >$ the polygon $A'B'C'H'D'E'$.

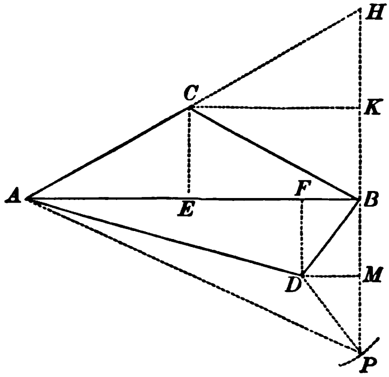
Take away from the two figures the equal $\triangle$s CHD and $C'H'D'$.

Then the polygon $ABCDE >$ the polygon $A'B'C'D'E'$.

Q. E. D.

PROPOSITION XXVI. THEOREM.

409. *Of all triangles having the same base and equal perimeters, the isosceles triangle is the maximum.*



Let the $\triangle$s ACB and ADB have equal perimeters, and let the $\triangle ACB$ be isosceles.

We are to prove $\triangle ACB > \triangle ADB$.

Draw the $\perp$s CE and DF.

$$\frac{\triangle ACB}{\triangle ABD} = \frac{CE}{DF}, \qquad \S\ 326$$

(*$\triangle$s having the same base are to each other as their altitudes*).

Produce AC to H, making $CH = AC$.

Draw HB.

The $\angle ABH$ is a rt. $\angle$, for it will be inscribed in the semicircle drawn from C as a centre, with the radius CB.

From C let fall the $\perp CK$;

and from D as a centre, with a radius equal to DB,

describe an arc cutting HB produced, at P.

Draw DP and AP,

and let fall the $\perp DM$.

Since $AH = AC + CB = AD + DB$,

and $AP < AD + DP$;

$\therefore AP < AD + DB$;

$\therefore AH > AP$.

$\therefore BH > BP$. § 56

Now $BK = \frac{1}{2} BH$, § 113

(*a* $\perp$ *drawn from the vertex of an isosceles* $\triangle$ *bisects the base*),

and $BM = \frac{1}{2} BP$. § 113

But $CE = BK$, § 135

(*||s comprehended between ||s are equal*);

and $DF = BM$, § 135

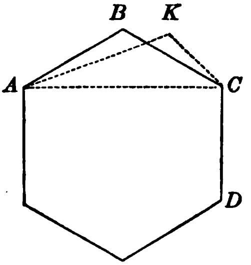
$\therefore CE > DF$.

$\therefore \triangle ACB > \triangle ADB$.

Q. E. D.

PROPOSITION XXVII. THEOREM.

410. *The maximum of isoperimetrical polygons of the same number of sides is equilateral.*



Let $ABCD$, etc., be the maximum of isoperimetrical polygons of any given number of sides.

We are to prove AB, BC, CD, *etc., equal.*

Draw AC.

The $\triangle ABC$ must be the maximum of all the ▲ which are formed upon AC with a perimeter equal to that of $\triangle ABC$.

Otherwise, a greater $\triangle AKC$ could be substituted for $\triangle ABC$, without changing the perimeter of the polygon.

But this is inconsistent with the hypothesis that the polygon $ABCD$, etc., is the maximum polygon.

$\therefore$ the $\triangle ABC$, is isosceles, § 409
(*of all* ▲ *having the same base and equal perimeters, the isosceles* $\triangle$ *is the maximum*).

In like manner it may be proved that $BC = CD$, etc.

Q. E. D.

411. COROLLARY. The maximum of isoperimetrical polygons of the same number of sides is a regular polygon.

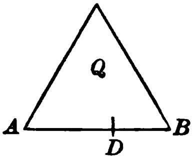
For, it is equilateral, § 410
(*the maximum of isoperimetrical polygons of the same number of sides is equilateral*).

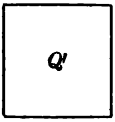
Also it can be inscribed in a ⊙, § 408
(*the maximum of all polygons formed of given sides can be inscribed in a* ⊙).

Hence it is regular, § 364
(*an equilateral polygon inscribed in a* ⊙ *is regular*).

PROPOSITION XXVIII. THEOREM.

412. *Of isoperimetrical regular polygons, that is greatest which has the greatest number of sides.*





Let Q be a regular polygon of three sides, and Q' be a regular polygon of four sides, each having the same perimeter.

We are to prove $Q' > Q$.

In any side $A\,B$ of Q, take any point D.

The polygon Q may be considered an irregular polygon of four sides, in which the sides $A\,D$ and $D\,B$ make with each other an ∠ equal to two rt. $\angle$s.

Then the irregular polygon Q, of four sides is less than the regular isoperimetrical polygon Q' of four sides, § 411

(the maximum of isoperimetrical polygons of the same number of sides is a regular polygon).

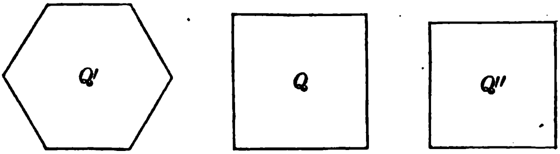
In like manner it may be shown that Q' is less than a regular isoperimetrical polygon of five sides, and so on.

Q. E. D.

413. COROLLARY. Of all isoperimetrical plane figures the circle is the maximum.

Proposition XXIX. Theorem.

414. *If a regular polygon be constructed with a given area, its perimeter will be the less the greater the number of its sides.*



Let Q and Q' be regular polygons having the same area, and let Q' have the greater number of sides.

We are to prove the perimeter of $Q >$ the perimeter of Q'.

Let Q'' be a regular polygon having the same perimeter as Q', and the same number of sides as Q.

Then Q' is $> Q''$, § 412

(*of isoperimetrical regular polygons, that is the greatest which has the greatest number of sides*).

But $Q = Q'$,

$\therefore$ Q is $> Q''$.

$\therefore$ the perimeter of Q is $>$ the perimeter of Q''.

But the perimeter of $Q' =$ the perimeter of Q'', Cons.

$\therefore$ the perimeter of Q is $>$ that of Q'.

Q. E. D.

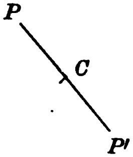
415. Corollary. The circumference of a circle is less than the perimeter of any other plane figure of equal area.

ON SYMMETRY. — SUPPLEMENTARY.

416. Two points are *Symmetrical* when they are situated on opposite sides of, and *at equal distances from*, a fixed point, line, or plane, taken as an object of reference.

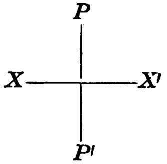
417. When a point is taken as an object of reference, it is called the *Centre of Symmetry;* when a line is taken, it is called the *Axis of Symmetry;* when a plane is taken, it is called the *Plane of Symmetry.*

418. *Two points* are symmetrical with respect to a *centre*, if the centre bisect the straight line terminated by these points. Thus, P, P' are symmetrical with respect to C, if C bisect the straight line PP'.

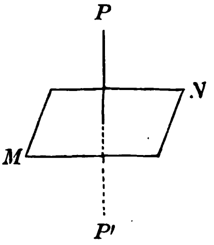


419. The distance of either of the two symmetrical points from the centre of symmetry is called the *Radius of Symmetry.* Thus either CP or CP' is the radius of symmetry.

420. *Two points* are symmetrical with respect to an *axis*, if the axis bisect at right angles the straight line terminated by these points. Thus, P, P' are symmetrical with respect to the axis XX', if XX' bisect PP' at right angles.



421. *Two points* are symmetrical with respect to a *plane*, if the plane bisect at right angles the straight line terminated by these points. Thus P, P' are symmetrical with respect to MN, if MN bisect PP' at right angles.



422. *Two plane figures* are symmetrical with respect to a *centre*, an *axis*, or a *plane*, if every point of either figure have its corresponding symmetrical point in the other.

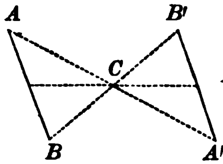


Fig. 1.

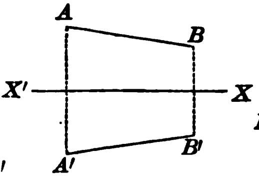


Fig. 2.

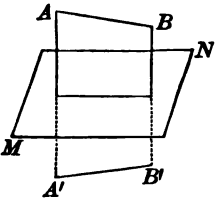


Fig. 3.

Thus, the lines $A B$ and $A' B'$ are symmetrical with respect to the centre C (Fig. 1), to the axis $X X'$ (Fig. 2), to the plane $M N$ (Fig. 3), if every point of either have its corresponding symmetrical point in the other.

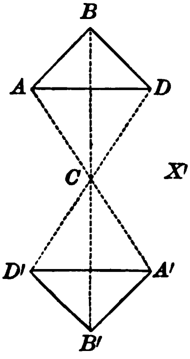


Fig. 4.

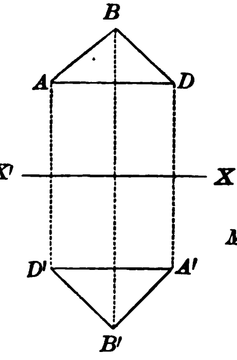


Fig. 5.

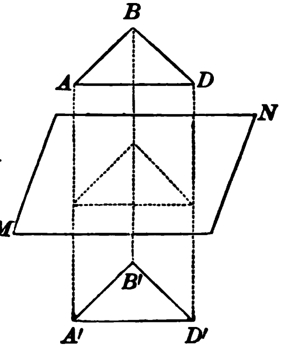


Fig. 6.

Also, the triangles $A B D$ and $A' B' D'$ are symmetrical with respect to the centre C (Fig. 4), to the axis $X X'$ (Fig. 5), to the plane $M N$ (Fig. 6), if every point in the perimeter of either have its corresponding symmetrical point in the perimeter of the other.

423. Def. In two symmetrical figures the corresponding symmetrical points and lines are called *homologous*.

Two symmetrical figures with respect to a centre can be brought into coincidence by revolving one of them in its own plane about the centre, every radius of symmetry revolving through two right angles at the same time.

Two symmetrical figures with respect to an axis can be brought into coincidence by the revolution of either about the axis until it comes into the plane of the other.

424. Def. A single figure is a *symmetrical figure,* either when it can be divided by an axis, or plane, into two figures symmetrical with respect to that axis or plane; or, when it has a centre such that every straight line drawn through it cuts the perimeter of the figure in two points which are symmetrical with respect to that centre.

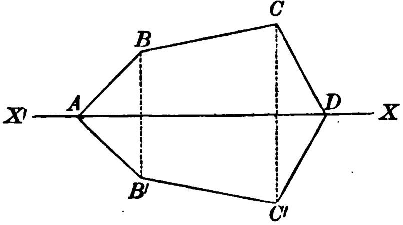


Fig. 1.

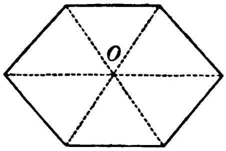


Fig. 2.

Thus, Fig. 1 is a symmetrical figure with respect to the axis XX', if divided by XX' into figures $ABCD$ and $AB'C'D$ which are symmetrical with respect to XX'.

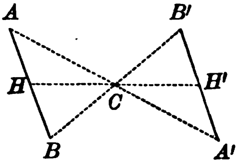
And, Fig. 2 is a symmetrical figure with respect to the centre O, if the centre O bisect every straight line drawn through it and terminated by the perimeter.

Every such straight line is called a *diameter.*

The circle is an illustration of a single figure symmetrical with respect to its centre as the *centre of symmetry,* or to any diameter as the *axis of symmetry.*

PROPOSITION XXX. THEOREM.

425. *Two equal and parallel lines are symmetrical with respect to a centre.*



Let $A\,B$ and $A'\,B'$ be equal and parallel lines.

We are to prove $A\,B$ *and* $A'\,B'$ *symmetrical.*

Draw $A\,A'$ and $B\,B'$, and through the point of their intersection C, draw any other line $H\,C\,H'$, terminated in $A\,B$ and $A'\,B'$.

In the $\triangle$ $C\,A\,B$ and $C\,A'\,B'$

$$A\,B = A'\,B', \qquad \text{Hyp.}$$

also, $\angle$ A and $B = \angle$ A' and B' respectively, § 68
(being alt.-int. $\angle$),

$$\therefore \triangle\, C\,A\,B = \triangle\, C\,A'\,B'; \qquad \S\ 107$$

$\therefore$ $C\,A$ and $C\,B = C\,A'$ and $C\,B'$ respectively,
(being homologous sides of equal $\triangle$).

Now in the $\triangle$ $A\,C\,H$ and $A'\,C\,H'$

$$A\,C = A'\,C,$$

$\angle$ A and $A\,C\,H = \angle$ A' and $A'\,C\,H'$ respectively,

$$\therefore \triangle\, A\,C\,H = \triangle\, A'\,C\,H', \qquad \S\ 107$$

(having a side and two adj. $\angle$ of the one equal respectively to a side and two adj. $\angle$ of the other).

$$\therefore C\,H = C\,H',$$

(being homologous sides of equal $\triangle$).

$\therefore$ H' is the symmetrical point of H.

But H is *any* point in $A\,B$;

$\therefore$ *every* point in $A\,B$ has its symmetrical point in $A'\,B'$.

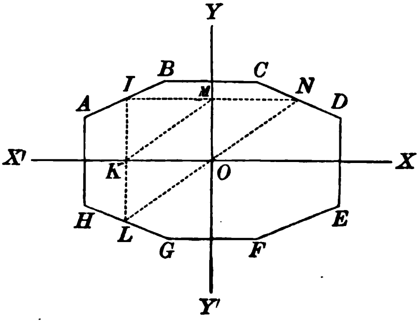
$\therefore$ $A\,B$ and $A'\,B'$ are symmetrical with respect to C as a centre of symmetry.

Q. E. D.

426. COROLLARY. If the extremities of one line be respectively the symmetricals of another line with respect to the same centre, the two lines are symmetrical with respect to that centre.

PROPOSITION XXXI. THEOREM.

427. *If a figure be symmetrical with respect to two axes perpendicular to each other, it is symmetrical with respect to their intersection as a centre.*



Let the figure $ABCDEFGH$ be symmetrical to the two axes XX', YY' which intersect at O.

We are to prove O the centre of symmetry of the figure.

Let I be any point in the perimeter of the figure.

Draw $IKL \perp$ to XX', and $IMN \perp$ to YY'.

Join LO, ON, and KM.

Now $KI = KL$, § 420

(the figure being symmetrical with respect to XX').

But $KI = OM$, § 135

(||s comprehended between ||s are equal).

$\therefore KL = OM$. Ax. 1

$\therefore KLOM$ is a ▱, § 136

(having two sides equal and parallel).

$\therefore LO$ is equal and parallel to KM, § 134

(being opposite sides of a ▱).

In like manner we may prove ON equal and parallel to KM.

Hence the points L, O, and N are in the same straight line drawn through the point O || to KM.

Also $LO = ON$,

(since each is equal to KM).

$\therefore$ any straight line LON, drawn through O, is bisected at O.

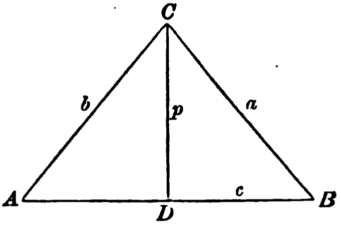
$\therefore O$ is the centre of symmetry of the figure. § 424

Q. E. D.

EXERCISES.

1. The area of any triangle may be found as follows: From half the sum of the three sides subtract each side severally, multiply together the half sum and the three remainders, and extract the square root of the product.

Denote the sides of the triangle ABC by a, b, c, the altitude by p, and $\frac{a+b+c}{2}$ by s.



Show that

$$a^2 = b^2 + c^2 - 2c \times AD,$$

$$AD = \frac{b^2 + c^2 - a^2}{2c};$$

and show that

$$p^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2},$$

$$p = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2c},$$

$$p = \frac{\sqrt{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}}{2c}.$$

Hence, show that area of $\triangle ABC$, which is equal to $\frac{c \times p}{2}$,

$$= \tfrac{1}{4}\sqrt{(b+c+a)(b+c-a)(a+b-c)(a-b+c)},$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

2. Show that the area of an equilateral triangle, each side of which is denoted by a, is equal to $\frac{a^2\sqrt{3}}{4}$.

3. How many acres are contained in a triangle whose sides are respectively 60, 70, and 80 chains?

4. How many feet are contained in a triangle each side of which is 75 feet?

BOOK VI.

PLANES AND SOLID ANGLES.

ON LINES AND PLANES.

428. **Def.** A *Plane* has already been defined as a surface such that the straight line joining any two points in it lies wholly in the surface.

The plane is considered to be indefinite in extent, so that however far the straight line be produced, all its points lie in the plane. A plane is usually represented by a quadrilateral supposed to lie in the plane.

429. **Def.** The *Foot* of a line is the point in which it meets the plane.

430. **Def.** A straight line is *perpendicular to a plane* if it be perpendicular to every straight line of the plane drawn through its foot.

In this case the plane is perpendicular to the line.

431. **Def.** The *Distance* from a point to a plane is the perpendicular distance from the point to the plane.

432. **Def.** A line is *parallel to a plane* if all its points be equally distant from the plane.

In this case the plane is parallel to the line.

433. **Def.** A line is *oblique to a plane* if it be neither perpendicular nor parallel to the plane.

434. **Def.** *Two planes are parallel* if all the points of either be equally distant from the other.

435. **Def.** The *Projection of a point* on a plane is the foot of the perpendicular from the point to the plane.

436. **Def.** The *projection of a line* on a plane is the locus of the projections of all its points.

437. **Def.** The plane embracing the perpendiculars which project the points of a straight line upon a plane is called the *projecting plane* of the line.

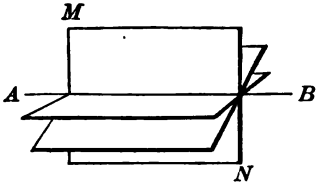
438. Def. The *angle* which a line makes with a plane is the angle which it makes with its projection on the plane.

This angle is called the *Inclination* of the line to the plane.

439. Def. A plane is *determined* by lines or points, if no other plane can embrace these lines or points without being coincident with that plane.

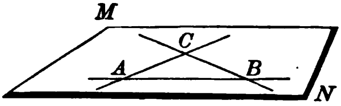
440. Def. The *intersection* of *two planes* is the locus of all the points common to the two planes.

441. *An infinite number of planes may embrace the same straight line.*



Thus, if the plane $M N$ embrace the line $A B$ it may be made to revolve about $A B$ as an axis, and to occupy an infinite number of positions, each of which is the position of a plane embracing the line $A B$.

442. *A plane is determined by a straight line and a point without that line.*



Thus, let any plane embracing the straight line $A B$ revolve about the line as an axis until it embraces the point C.

Now if the plane revolve either way about the line $A B$ as an axis, it will cease to embrace the point C.

Hence any other plane embracing the line $A B$ and the point C must be coincident with the first plane. § 439

443. *Three points not in a straight line determine a plane.*

For, by joining any two of the points, we have a straight line and a point which determine a plane. § 442

444. *Two intersecting straight lines determine a plane.*

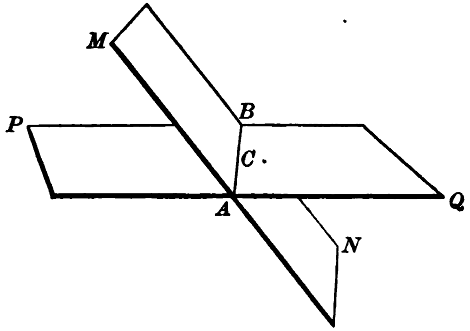
For, a plane embracing one of these straight lines and any point of the other line (except the point of intersection) is determined. § 442

445. *Two parallel straight lines determine a plane.*

For, a plane embracing either of these parallels and any point in the other is determined. § 442

PROPOSITION I. THEOREM.

446. *If two planes cut one another their intersection is a straight line.*



Let MN and PQ be two planes which cut one another.

We are to prove their intersection a straight line.

Let A and B be two points common to the two planes.

Draw the straight line AB.

Since the points A and B are common to the two planes, the straight line AB lies in both planes. § 428

Now, no point out of this line can be in both planes; for, if it be possible, let C be such a point.

But there can be but *one* plane embracing the point C and the line AB. § 442

$\therefore$ C does not lie in *both* planes.

$\therefore$ every point in the intersection of the two planes lies in the straight line AB.

Q. E. D.

PROPOSITION II. THEOREM.

447. *From a point without a plane only one perpendicular can be drawn to the plane; and at a given point in a plane only one perpendicular can be erected to the plane.*

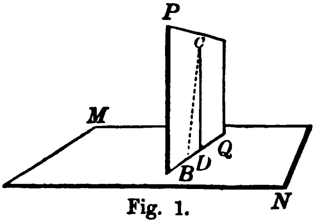


Fig. 1.

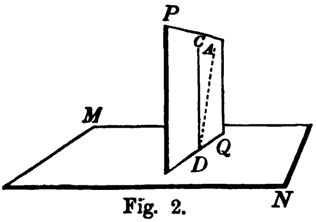


Fig. 2.

Let CD (*Fig. 1*) be a perpendicular let fall from the point C to the plane MN.

We are to prove that no other ⊥ can be drawn from the point C to the plane MN.

If it be possible, let CB be another ⊥ to the plane MN,

and let a plane PQ pass through the lines CB and CD.

The intersection of PQ with the plane MN is a straight line BD. § 446

Now if CD and CB be both ⊥ to the plane, the △ CBD would have two rt. ⦞, CBD and CDB, which is impossible. § 102

Let DC (*Fig. 2*) be a perpendicular to the plane MN at the point D.

If it be possible, let DA be another ⊥ to the plane from the point D,

and let a plane PQ pass through the lines DC and DA.

The intersection of PQ with the plane MN is a straight line.

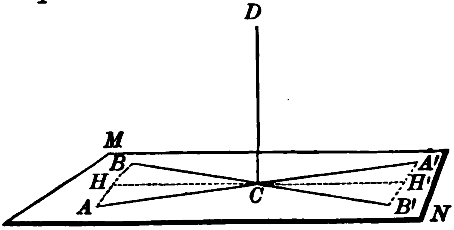
Now if DC and DA could both be ⊥ to the plane MN at D, we should have in the plane PQ two straight lines ⊥ to the line DQ at the point D, which is impossible. § 61

Q. E. D.

448. COROLLARY. A perpendicular is the shortest distance from a point to a plane.

PROPOSITION III. THEOREM.

449. *If a straight line be perpendicular to each of two straight lines drawn through its foot in a plane it is perpendicular to the plane.*



Let DC be perpendicular to each of the two lines ACA' and BCB' drawn through its foot in the plane MN.

We are to prove $DC \perp$ *to the plane* MN.

Take $CA = CA'$ and $CB = CB'$.

Join AB and $A'B'$.

Then AB and $A'B'$ are symmetrical with respect to C, § 426
(their extremities being symmetrical).

Through C draw any line HCH' in the plane MN.

Then H and H' are symmetrical, § 422
(being corresponding points in the symmetrical lines AB and $A'B'$).

About C, the centre of symmetry, revolve AB, keeping AC and $BC \perp$ to CD, until it comes into coincidence with $A'B'$.

Then the point H will coincide with its symmetrical point H',

and $\angle DCH$ will coincide with, and be equal to, $\angle DCH'$.

$\therefore$ ∠s DCH and DCH' are rt. ∠s. § 25

$\therefore DC$ is $\perp$ to HCH'.

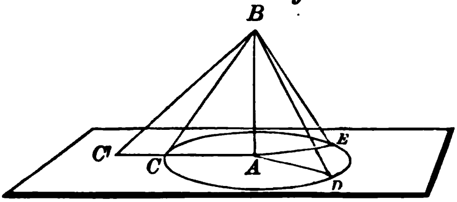
Now since DC is $\perp$ to *any* line, HCH', drawn through its foot in the plane MN, it is $\perp$ to *every* such line.

$\therefore DC$ is $\perp$ to the plane MN. § 430

Q. E. D.

PROPOSITION IV. THEOREM.

450. *Oblique lines drawn from a point to a plane at equal distances from the foot of the perpendicular are equal; and of two oblique lines unequally distant from the foot of the perpendicular the more remote is the greater.*



Let the oblique lines BC, BD, and BE, be drawn at equal distances, AC, AD, and AE, from the foot of the perpendicular BA; and let BC' be drawn more remote from the foot of the perpendicular than BC.

We are to prove I. $BC = BD = BE$.

II. $BC' > BC$.

I. In the rt. $\triangle$s BAC and BAD

$BA = BA$, Iden.

$AC = AD$, Hyp.

and rt. $\angle BAC =$ rt. $\angle BAD$.

$\therefore \triangle BAC = \triangle BAD$, § 106

$\therefore BC = BD$,

(being homologous sides of equal $\triangle$s).

II. Since AC' is $> AC$,

BC' is $> BC$, § 55

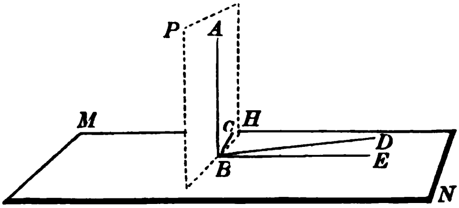
Q. E. D.

451. COROLLARY 1. Equal oblique lines from a point to a plane meet the plane at equal distances from the perpendicular; and of two unequal oblique lines, the greater meets the plane at the greater distance from the perpendicular.

452. COR. 2. All equal oblique lines BC, BD, etc., drawn from a point to a plane terminate in the circumference CDE described from A as a centre with a radius equal to AC. Hence, to draw a perpendicular from a point to a plane, draw any oblique line from the given point to the plane; revolve this line about the point, tracing the circumference of a circle in the plane, and draw a line from the point to the centre of the circle.

PROPOSITION V. THEOREM.

453. *If three straight lines meet at one point, and a straight line be perpendicular to each of them at that point, the three straight lines lie in the same plane.*



Let the straight line $A B$ be perpendicular to each of the straight lines $B C$, $B D$, and $B E$, at B.

We are to prove $B C$, $B D$, and BE in the same plane $M N$.

If not, let $B D$ and BE be in the plane $M N$, and $B C$ without it; and let $P H$, passing through $A B$ and $B C$, cut the plane $M N$ in the straight line $B H$.

Now $A B$, $B C$, and $B H$ are all in the plane $P H$,

and since $A B$ is ⊥ to $B D$ and $B E$, it is ⊥ to the plane $M N$, § 449

(if a straight line be ⊥ to each of two straight lines drawn through its foot in a plane, it is ⊥ to the plane).

∴ $A B$ is ⊥ to $B H$, a straight line in the plane $M N$, § 430

(a ⊥ to a plane is ⊥ to every straight line in that plane drawn through its foot).

That is $\angle A BH$ is a rt. ∠.

But $\angle A B C$ is a rt. ∠. Hyp.

$\therefore \angle A B C = \angle A B H.$

∴ $B C$ and $B H$ coincide.

∴ $B C$ is not without the plane $M N$.

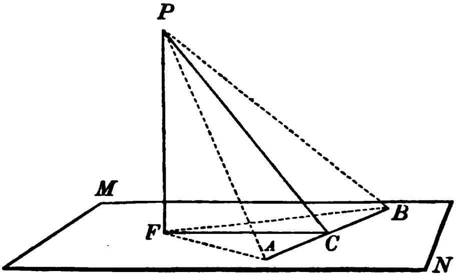
Q. E. D

454. COROLLARY. The locus of all perpendiculars to a given straight line at a given point is a plane perpendicular to this given straight line at the given point.

455. SCHOLIUM. In the geometry of space the term *locus* has the same signification as in plane geometry, only it is not limited to lines, but is extended to include surfaces.

PROPOSITION VI. THEOREM.

456. *If from the foot of a perpendicular to a plane a straight line be drawn at right angles to any line of the plane, the line drawn from its intersection with the line of the plane to any point of the perpendicular is perpendicular to the line of the plane.*



Let PF be a perpendicular to the plane MN, FC a perpendicular from the foot of PF to any line AB, in the plane MN, and CP a line drawn from its intersection with AB to any point P in the perpendicular PF.

We are to prove $CP \perp$ *to* AB.

Take $CA = CB$ and draw FA, FB, PA, PB.

Now $FA = FB$, § 53

(two oblique lines drawn from a point in a ⊥ cutting off equal distances from the foot of the ⊥ are equal),

and $PA = PB$, § 450

(oblique lines drawn from a point to a plane at equal distances from the foot of the ⊥ are equal).

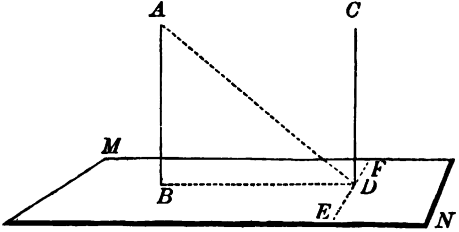
$\therefore PC$ is $\perp$ to AB, § 60

(two points equally distant from the extremities of a straight line determine the ⊥ at the middle point of the line).

Q. E. D.

PROPOSITION VII. THEOREM.

457. *If a line be perpendicular to a plane, every line which is parallel to this perpendicular is likewise perpendicular to the plane.*



Let AB be perpendicular to the plane MN, and CD any line parallel to AB.

We are to prove CD perpendicular to the plane MN.

Draw BD in the plane MN, and through D draw EF in the plane MN ⊥ to BD, and join D with any point in AB, as A.

BD is ⊥ to AB, § 430

(*if a straight line be ⊥ to a plane it is ⊥ to every line of the plane drawn through its foot*);

it is also ⊥ to CD, § 67

(*if a straight line be ⊥ to one of two* ||*s, it is ⊥ to the other*).

Now EF is ⊥ to AD, § 456

(*if from the foot of a ⊥ to a plane a straight line be drawn at right angles to any line of the plane, the line drawn from its intersection with the line of the plane to any point in the ⊥ is ⊥ to the line of the plane*),

and is also ⊥ to BD. Cons.

∴ EF is ⊥ to the plane $ABDC$, § 449

(*a straight line ⊥ to two straight lines drawn through its foot in a plane is ⊥ to the plane*),

∴ EF is ⊥ to CD, § 430

(*if a straight line be ⊥ to a plane it is ⊥ to every line of the plane drawn through its foot*).

∴ CD is ⊥ to BD and EF, and consequently to the plane MN. § 449

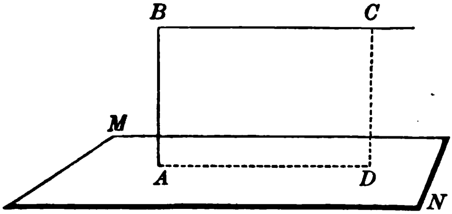
Q. E. D.

458. COROLLARY 1. Two lines which are perpendicular to the same plane are parallel.

459. COR. 2. Two lines parallel to a third straight line not in their own plane are parallel to each other.

PROPOSITION VIII. THEOREM.

460. *If a straight line and a plane be perpendicular to the same straight line, they are parallel.*



Let the straight line BC and the plane MN be perpendicular to the straight line AB.

We are to prove $BC \parallel$ *to* MN.

From any point C of the line BC let CD be drawn perpendicular to MN.

Join AD.

BA and CD are parallel, § 458

(*two straight lines* ⊥ *to the same plane are* ‖).

AD is ⊥ to BA, § 430

(*if a straight line be* ⊥ *to a plane it is* ⊥ *to every line of the plane drawn through its foot*).

∴ AD and BC are parallel, § 65

(*two straight lines* ⊥ *to the same straight line are* ‖).

∴ $ABCD$ is a ▱. § 125

∴ $CD = AB$. § 134

Now, since C is *any* point in the line BC, *all* the points in BC are equally distant from the plane MN.

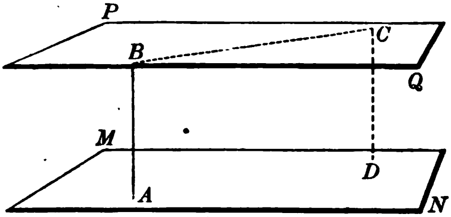
∴ BC is ‖ to MN, § 432

(*a line is* ‖ *to a plane if all its points be equally distant from the plane*).

Q. E. D.

PROPOSITION IX. THEOREM.

461. *If two planes be perpendicular to the same straight line they are parallel.*



Let the two planes *M N* and *P Q* be perpendicular to the straight line *A B*.

We are to prove *P Q* ‖ *to M N.*

From any point C in the plane PQ draw $CD \perp$ to MN.

Join BC.

BC is $\perp$ to AB, § 430

(if a straight line be $\perp$ to a plane it is $\perp$ to every line of the plane drawn through its foot).

$\therefore BC$ is ‖ to the plane MN, § 460

(if a straight line and a plane be $\perp$ to the same straight line they are ‖).

$\therefore CD$ is equal to AB, § 432

(if a straight line be ‖ to a plane, all its points are equally distant from the plane).

Since C is *any* point in the plane PQ, *all* the points in the plane PQ are at equal distances from MN.

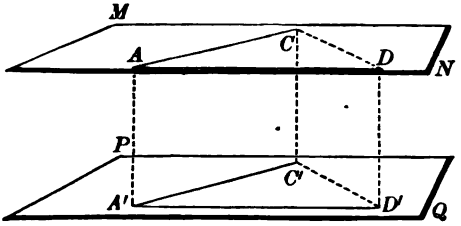
$\therefore PQ$ is ‖ to MN, § 434

(two planes are ‖ if all the points of either be equally distant from the other).

Q. E. D.

PROPOSITION X. THEOREM.

462. *If two angles not in the same plane have their sides respectively parallel and lying in the same direction, they are equal.*



Let $\angle$s A and A' be respectively in the planes $M\,N$ and $P\,Q$ and have $A\,D$ parallel to $A'\,D'$ and $A\,C$ parallel to $A'\,C'$ and lying in the same direction.

We are to prove $\angle A = \angle A'$.

Take $A\,D = A'\,D'$ and $A\,C = A'\,C'$.

Join $A\,A'$, $D\,D'$, $C\,C'$, $C\,D$, $C'\,D'$.

Since $A\,D$ is equal and $\parallel$ to $A'\,D'$, the figure $A\,D\,D'\,A'$ is a ▱, § 136

$\therefore A\,A' = D\,D'$. § 134

In like manner $A\,A' = C\,C'$,

$\therefore C\,C' = D\,D'$. Ax. 1

Also, since $C\,C'$ and $D\,D'$ are respectively $\parallel$ to $A\,A'$, they are $\parallel$ to each other, § 459

(*two straight lines $\parallel$ to a third straight line not in their own plane are $\parallel$ to each other*).

$\therefore C\,D\,D'\,C'$ is a ▱. § 136

$\therefore C\,D = C'\,D'$, § 134

$\therefore \triangle A\,D\,C = \triangle A'\,D'\,C'$, § 108

(*having three sides of the one equal respectively to three sides of the other*).

$\therefore \angle A = \angle A'$,

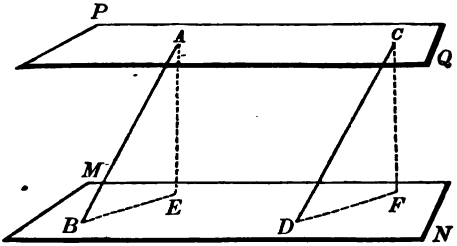
(*being homologous $\angle$s of equal $\triangle$s*).

Q. E. D.

463. COROLLARY. If two angles lie in different planes and have their sides parallel and extending in the same direction, the planes are parallel. For the intersecting lines, $A\,C$ and $A\,D$, which determine the plane $M\,N$ are parallel respectively to the lines $A'\,C'$ and $A'\,D'$ which determine the plane $P\,Q$, therefore the planes are determined parallel.

PROPOSITION XI. THEOREM.

464. *Two parallel lines comprehended between two parallel planes are equal.*



Let the two parallel lines AB and CD be included between the parallel planes MN and PQ.

We are to prove $AB = CD$.

If AB and CD be ⊥ to the two ∥ planes they are equal, § 434
(*if two planes be ∥, all the points of either are equally distant from the other*).

If AB and CD be not ⊥ to the two ∥ planes, draw from the points A and C the lines AE and CF ⊥ to the plane MN.

AE is ∥ to CF, § 458
(*two lines ⊥ to the same plane are ∥*).

Join BE and DF.

In $\triangle$s AEB and CFD,

$AE = CF$, § 434

$\angle AEB = \angle CFD$, § 430
(*if a straight line be ⊥ to a plane it is ⊥ to any line of the plane drawn through its foot*);

and $\angle BAE = \angle DCF$, § 462
(*if two* $\angle$s *not in the same plane have their sides ∥ and lying in the same direction they are equal*).

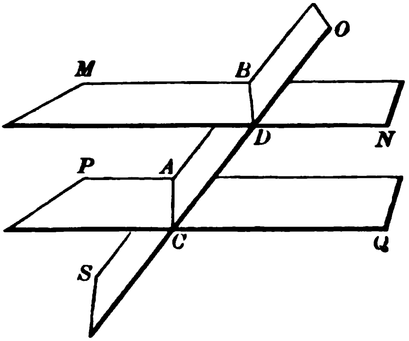
$\therefore \triangle AEB = \triangle CFD$, § 107
(*having a side and two adj.* $\angle$s *of the one equal respectively to a side and two adj.* $\angle$s *of the other*).

Hence $AB = CD$,
(*being homologous sides of equal* $\triangle$s).

Q. E. D.

PROPOSITION XII. THEOREM.

465. *The intersections of two parallel planes by a third plane are parallel lines.*



Let the plane OS intersect the parallel planes PQ and MN in the lines AC and BD respectively.

We are to prove $AC \parallel$ to BD.

Through the points A and C draw the ∥ lines AB and CD in the plane OS.

Now $AB = CD$, § 464

(∥ *lines comprehended between* ∥ *planes are equal*).

$\therefore ABCD$ is a ▱, § 136

(*having two sides equal and* ∥).

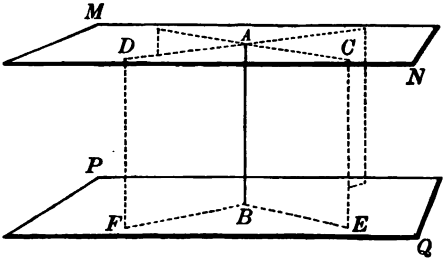
$\therefore AC$ is ∥ to BD, § 125

(*being opposite sides of a* ▱).

Q. E. D.

PROPOSITION XIII. THEOREM.

466. *If a straight line be perpendicular to one of two parallel planes it is perpendicular to the other.*



Let MN and PQ be parallel planes and AB be perpendicular to PQ.

We are to prove $AB \perp$ to MN.

Let two planes embracing AB intersect the planes MN and PQ in AC, BE and AD, BF respectively.

Then AC is ‖ to BE and AD to BF, § 465

(*the intersections of two ‖ planes by a third plane are ‖ lines*).

But EB and FB are $\perp$ to AB, § 430

(*if a straight line be $\perp$ to a plane it is $\perp$ to every straight line of the plane drawn through its foot*).

∴ AC and AD which are respectively ‖ to BE and BF are $\perp$ to AB, § 67

(*if a straight line be $\perp$ to one of two ‖ lines, it is $\perp$ to the other*).

∴ AB is $\perp$ to MN, § 449

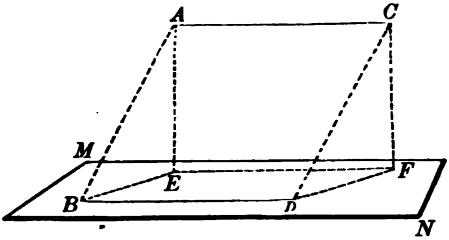
(*if a line be $\perp$ to two straight lines in a plane drawn through its foot it is $\perp$ to the plane*).

Q. E. D.

467. COROLLARY. If two planes be parallel to a third plane they are parallel to each other. For, every line perpendicular to this third plane is perpendicular to the other planes; and two planes perpendicular to a straight line are parallel.

PROPOSITION XIV. THEOREM.

468. *If a straight line be parallel to another straight line drawn in a plane, it is parallel to the plane.*



Let AC be parallel to the line BD in the plane MN.

We are to prove AC ∥ to the plane MN.

From A and C, any two points in AC, draw AB and CD $\perp$ to AC, and AE and CF $\perp$ to the plane MN.

Join BE and DF.

Now AB is ∥ to CD, § 65

(*two straight lines ⊥ to the same line are* ∥).

Also $AB = CD$, § 135

(∥ *lines comprehended between* ∥ *lines are equal*),

and AE is ∥ to CF, § 458

(*two straight lines ⊥ to the same plane are* ∥).

$\therefore \angle BAE = \angle DCF$, § 462

(*if two* ∠s *not in the same plane have their sides* ∥ *and lying in the same direction, they are equal*).

$\therefore$ rt. $\triangle AEB =$ rt. $\triangle CFD$, § 110

(*two rt.* △s *are equal when an acute* ∠ *and the hypotenuse of the one are equal respectively to an acute* ∠ *and the hypotenuse of the other*).

$\therefore AE = CF$,

(*being homologous sides of equal* △s).

Now since the points A and C, any two points in the line AC, are equally distant from the plane MN,

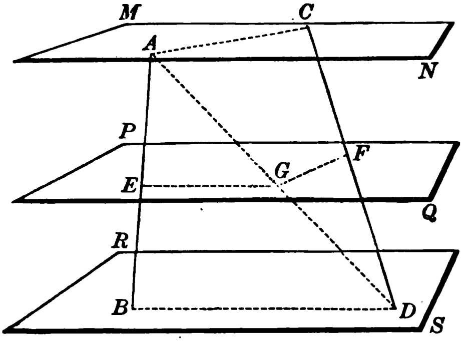
all the points in AC are equally distant from the plane MN.

$\therefore AC$ is ∥ to the plane MN. § 432

Q. E. D.

PROPOSITION XV. THEOREM.

469. *If two straight lines be intersected by three parallel planes their corresponding segments are proportional.*



Let AB and CD be intersected by the parallel planes MN, PQ, RS, in the points A, E, B, and C, F, D.

We are to prove $\frac{AE}{EB} = \frac{CF}{FD}$.

Draw AD cutting the plane PQ in G.

Join EG and FG.

Then EG is ∥ to BD, § 465

(*the intersections of two* ∥ *planes by a third plane are* ∥ *lines*).

$$\therefore \frac{AE}{EB} = \frac{AG}{GD},$$ § 275

(*a line drawn through two sides of a* △ ∥ *to the third side divides those sides proportionally*).

Also, GF is ∥ to AC, § 465

$$\therefore \frac{CF}{FD} = \frac{AG}{GD},$$ § 275

$$\therefore \frac{AE}{EB} = \frac{CF}{FD}.$$ Ax. 1

Q. E. D.

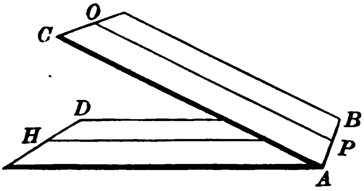
ON DIHEDRAL ANGLES.

470. DEF. The amount of rotation which one of two intersecting planes must make about their intersection in order to coincide with the other plane is called the *Dihedral* angle of the planes.

The *Faces* of a dihedral angle are the intersecting planes.

The *Edge* of a dihedral angle is the intersection of its faces.

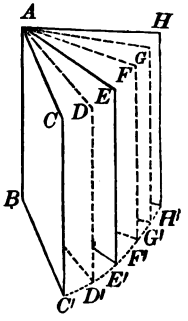
The *Plane* angle of a dihedral angle is the plane angle formed by two straight lines, one in each plane, perpendicular to the edge at the same point.



Thus, in the diagram, *C-A B-D* is a dihedral angle, *C B* and *D A* are its faces, *A B* is its edge, *O P H* is its plane angle if *O P* and *H P* in the faces be perpendicular to the edge *A B* at the same point *P*.

471. *The plane angle of a dihedral angle has the same magnitude from whatever point in the edge we draw the perpendiculars.* For every pair of such angles have their sides respectively parallel (§ 65), and hence are equal (§ 462).

Two equal dihedral angles, D-A B-C′, and D-A B-E′, have corresponding equal plane angles, D A C and D A E. This may be shown by superposition.



Any two dihedral angles, C-A B-E′ and E-A B-H′, have the same ratio as their corresponding plane angles, C A E and E A H. This may be shown by the method employed in § 200 and § 201.

Hence a dihedral angle is *measured* by its plane angle.

It must be observed that the sides of the plane angle which measures the dihedral angle must be perpendicular to the edge. Thus in the rectangular solid *A H*, Fig. 1, the dihedral angle *F-B A-H*, is a right dihedral angle, and is measured by the angle *C E D*, if its sides *C E* and *E D*, drawn in the planes *A F* and *A G* respectively, be perpendicular to *A B*. But angle *C′ E′ D′*, drawn as represented in the diagram, is acute, while angle *C″ E″ D″*, drawn as represented, is obtuse.

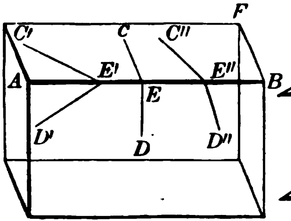


Fig. 1.

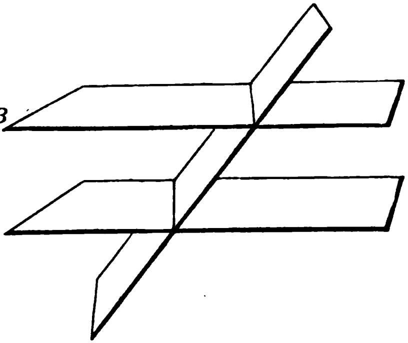


Fig. 2.

Many properties of dihedral angles can be established which are analogous to propositions relating to plane angles. Let the student prove the following:

1. If two planes intersect each other, their vertical dihedral angles are equal.

2. If a plane intersect two parallel planes, the exterior-interior dihedral angles are equal; the alternate-interior dihedral angles are equal; the two interior dihedral angles on the same side of the secant plane are supplements of each other.

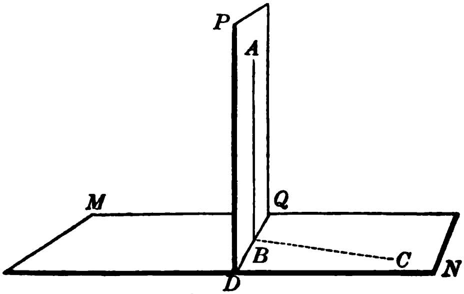
3. When two planes are cut by a third plane, if the exterior-interior dihedral angles be equal, or the alternate dihedral angles be equal, or the two interior dihedral angles on the same side of the secant plane be supplements of each other, the two planes are parallel.

4. Two dihedral angles are equal if their faces be respectively parallel and lie in the same direction, or opposite directions, from the edges.

5. Two dihedral angles are supplements of each other if two of their faces be parallel and lie in the same direction, and the other faces be parallel and lie in the opposite direction, from the edges.

PROPOSITION XVI. THEOREM.

472. *If a straight line be perpendicular to a plane every plane embracing the line is perpendicular to that plane.*



Let AB be perpendicular to the plane MN.

We are to prove any plane, PQ, embracing AB, perpendicular to MN.

At B draw, in the plane MN, $BC \perp$ to the intersection DQ.

Since AB is $\perp$ to MN, it is $\perp$ to DQ and BC, § 430

(*if a straight line be $\perp$ to a plane, it is $\perp$ to every straight line in that plane drawn through its foot*).

Now $\angle ABC$ is the measure of the dihedral $\angle P\text{-}DQ\text{-}N$. § 470

But $\angle ABC$ is a right angle,

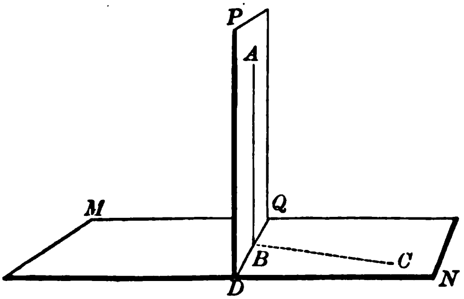
$\therefore$ the $\angle P\text{-}DQ\text{-}N$ is a right dihedral,

$\therefore PQ$ is $\perp$ to MN.

Q. E. D.

PROPOSITION XVII. THEOREM.

473. *If two planes be perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other plane.*



Let the planes MN and PQ be perpendicular to each other, and at any point B of their intersection DQ let BA be drawn in the plane PQ, perpendicular to DQ.

We are to prove $AB \perp$ *to the plane* MN.

Draw BC in the plane $MN \perp$ to DQ.

Then $\angle ABC$ is a right angle,

(being the plane $\angle$ of the rt. dihedral $\angle$ formed by the two planes).

$\therefore AB$ is $\perp$ to the two straight lines DQ and BC.

$\therefore AB$ is $\perp$ to the plane MN, § 449

(if a straight line be $\perp$ to two straight lines drawn through its foot in a plane, it is $\perp$ to the plane).

Q. E. D.

Proposition XVIII. Theorem.

474. *If two planes be perpendicular to each other, a straight line drawn through any point of intersection perpendicular to one of the planes will lie in the other plane.*

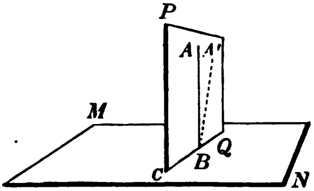


Fig. 1.

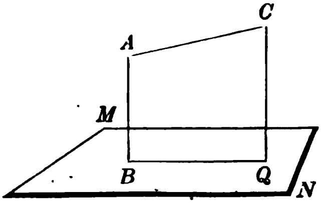


Fig. 2.

Let PQ (Fig. 1) be perpendicular to the plane MN, CQ their intersection, and BA be drawn through any point B in CQ perpendicular to the plane MN.

We are to prove that BA lies in the plane PQ.

At the point B draw BA' in the plane PQ $\perp$ to the intersection CQ.

The line BA' will be $\perp$ to the plane MN, § 472

(*if two planes be $\perp$ to each other, a straight line drawn in one of them $\perp$ to their intersection is $\perp$ to the other*).

Now BA is $\perp$ to the plane MN; Hyp.

$\therefore$ BA and BA' coincide, § 447

(*at a given point in a plane only one $\perp$ can be erected to that plane*).

But BA' lies in the plane PQ;

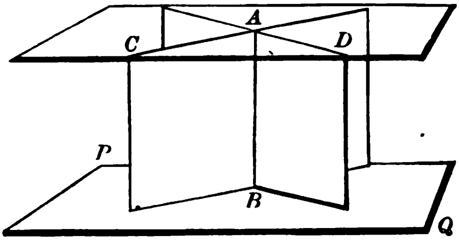
$\therefore$ BA, which coincides with BA', lies in the plane PQ.

Q. E. D.

Scholium. Through a line parallel or oblique to a plane, as AC, Fig. 2, only one plane can be passed perpendicular to the given plane.

PROPOSITION XIX. THEOREM.

475. *If two intersecting planes be each perpendicular to a third plane, their intersection is also perpendicular to that plane.*



Let the planes BD and BC intersecting in the line AB be perpendicular to the plane PQ.

We are to prove $AB \perp$ *to the plane* PQ.

A perpendicular erected at B, a point common to the three planes, will lie in the two planes BC and BD, § 473

(*if two planes be* $\perp$ *to each other, a straight line drawn through any point of intersection* $\perp$ *to one of the planes will lie in the other plane*).

And, since this $\perp$ lies in both the planes, BC and BD, it must coincide with their intersection.

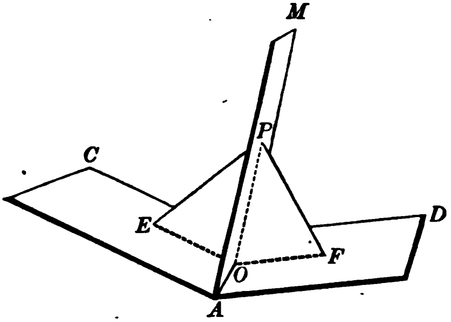
$\therefore AB$ is $\perp$ to the plane PQ.

Q. E. D.

476. COROLLARY. If a plane be perpendicular to each of two intersecting planes, it is perpendicular to the intersection of those planes.

PROPOSITION XX. THEOREM.

477. *Every point in the plane which bisects a dihedral angle is equally distant from the faces of that angle.*



Let plane AM bisect the dihedral angle formed by the planes AD and AC; and let PE and PF be perpendiculars drawn from any point P in the plane AM to the planes AC and AD.

We are to prove $PE = PF$.

Through PE and PF pass a plane intersecting the planes AC and AD in OE and OF.

Join PO.

Now the plane PEF is ⊥ to each of the planes AC and AD, § 471

(if a straight line be ⊥ to a plane, any plane embracing the line is ⊥ to that plane);

∴ the plane PEF is ⊥ to their intersection AO. § 476

(If a plane be ⊥ to each of two intersecting planes, it is ⊥ to the intersection of these planes).

$$\therefore \angle POE = \angle POF,$$

(being measures respectively of the equal dihedral ∠s M-OA-C and M-OA-D).

∴ rt. △ POE = rt. △ POF, § 110

$$\therefore PE = PF,$$

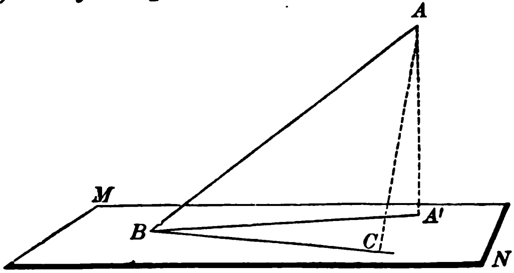
(being homologous sides of equal △s).

Q. E. D.

SUPPLEMENTARY PROPOSITIONS.

PROPOSITION XXI. THEOREM.

478. *The acute angle which a straight line makes with its own projection on a plane is the least angle which it makes with any line of that plane.*



Let BA meet the plane MN at B, and let BA' be its projection upon the plane MN, and BC any other line drawn through B in the plane.

We are to prove $\angle ABA' < \angle ABC$.

Take $BC = BA'$.

Join AC.

In the $\triangle$ ABA' and ABC,

$AB = AB$, Iden.

$BA' = BC$, Cons.

but $AA' < AC$, § 448

(*a* $\perp$ *is the shortest distance from a point to a plane*).

$\therefore \angle ABA' < \angle ABC$, § 116

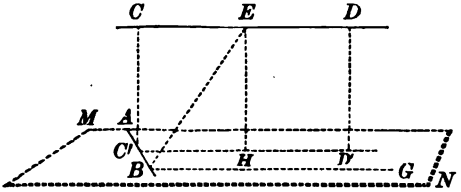
(*if two sides of a* $\triangle$ *be equal respectively to two sides of another, but the third side of the first* $\triangle$ *be greater than the third side of the second, then the* $\angle$ *opposite the third side of the first* $\triangle$ *is greater than the* $\angle$ *opposite the third side of the second*).

Q. E. D.

EXERCISE. — The angle included by two perpendiculars drawn from any point within a dihedral angle to its faces, is the supplement of the dihedral angle.

PROPOSITION XXII. THEOREM.

479. *If two straight lines be not in the same plane, one and only one common perpendicular to the lines can be drawn.*



Let AB and CD be two given straight lines not in the same plane.

We are to prove one and only one common perpendicular to the two lines can be drawn.

Since AB and CD are not in the same plane they are not ∥, § 474

(*two* ∥*s lie in the same plane*).

Through the line AB pass the plane MN ∥ to CD.

Since CD is ∥ to the plane MN, all its points are equally distant from the plane MN; § 432

hence $C'D'$, the projection of the line CD on the plane MN, will be ∥ to CD, § 76

and will intersect the line AB at some point as C'.

Now since CC' is the line which projects the point C upon the plane MN, it is ⊥ to the plane MN; § 435

hence CC' is ⊥ to $C'D'$ and AB, § 430

(*if a line be ⊥ to a plane, it is ⊥ to every line drawn through its foot in the plane*).

Also, CC' is ⊥ to CD, § 67

∴ CC' is the common ⊥ to the lines CD and AB.

Moreover, line CC' is the only common ⊥.

For, if another line EB, drawn between AB and CD, could be ⊥ to AB and CD, it would also be ⊥ to a line BG drawn ∥ to CD in the plane MN, § 67

and hence ⊥ to the plane MN. § 449

But EH, drawn in the plane CD' ∥ to CC', is ⊥ to the plane MN. § 457

Hence we should have two ⊥s from the point E to the plane MN, which is impossible, § 441

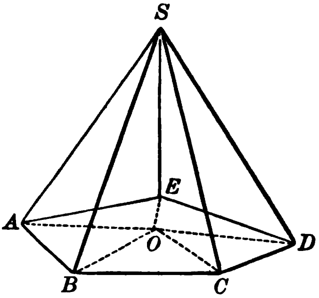
∴ CC' is the only common ⊥ to the lines CD and AB.

Q. E. D.

ON POLYHEDRAL ANGLES.

480. DEF. A *Polyhedral* angle is the *extent of opening* of three or more planes meeting in a common point.

Thus the figure *S-A B C D E*, formed by the planes *A S B*, *B S C*, *C S D*, *D S E*, *E S A*, meeting in the common point *S*, is a polyhedral angle.



The point *S* is the *vertex* of the angle.

The intersections of the planes *S A*, *S B*, etc., are its *edges*.

The portions of the planes bounded by the edges are its *faces*.

The plane angles *A S B*, *B S C*, etc., formed by the edges are its *face angles*.

481. DEF. Polyhedral angles are classified as *trihedral*, *quadrahedral*, etc., according to the number of the faces.

482. DEF. Trihedral angles are *rectangular*, *bi-rectangular*, or *tri-rectangular*, according as they have one, two, or three right dihedral angles.

483. DEF. Trihedral angles are *scalene*, *isosceles*, or *equilateral*, according as the face angles are all unequal, two equal, or three equal.

484. DEF. A polyhedral angle is *convex*, if the polygon formed by the intersections of a plane with all its faces be a convex [illegible]on.

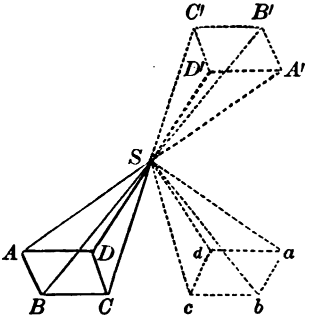
485. DEF. Two polyhedral angles are *equal* when they can be applied to each other so as to coincide in all their parts.

Since two equal polyhedral angles coincide however far their edges and faces be produced, the *magnitude* of a polyhedral angle does not depend upon the extent of its faces. But, in order to represent the angle in a diagram, it is usual to pass a plane, as

$ABCDE$, cutting all its faces in the straight lines, AB, BC, etc.; and by the face ASB is meant the indefinite surface included between the lines SA and SB indefinitely produced.

486. Def. Two polyhedral angles are ***symmetrical*** if they have the same number of faces, and the successive dihedral and face angles respectively equal but arranged in reverse order.

Thus, if the edges AS, BS, etc., of the polyhedral angle, $S\text{-}ABCD$, be produced, there is formed another polyhedral angle, $S\text{-}A'B'C'D'$, which is symmetrical with the first, the vertex S being the centre of symmetry.



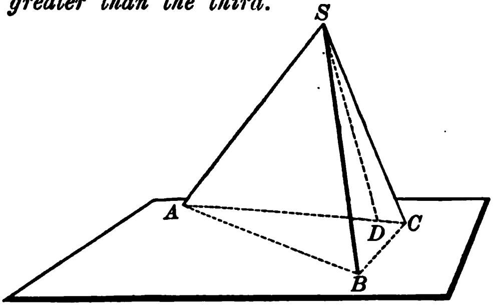
If we take $SA' = SA$, and through the points A and A' the parallel planes $ABCD$ and $A'B'C'D'$ be passed, we shall have $SB' = SB$, $SC' = SC$, etc. For if we conceive a third parallel plane to pass through S, then AA', BB', etc., are divided proportionally, § 469. And if any one of them be bisected at S, the others are also bisected at S. Hence, the points A', B', etc., are symmetrical with A, B, etc.

Moreover, the two symmetrical polyhedral angles are equal in all their parts. For their face angles ASB and $A'SB'$, BSC and $B'SC'$ are equal each to each, being vertical plane angles. And the dihedral angles formed at the edges SA and SA', SB and SB', are equal each to each, being vertical dihedral angles.

Now if the polyhedral angle $S\text{-}A'B'C'D'$ be revolved about the vertex S until the polygon $A'B'C'D'$ is brought into [illegible] position $abcd$, in the same plane with $ABCD$, it will be evident that while the parts ASB, BSC, etc., succeed each other in the order from *left to right*, the corresponding equal parts aSb, bSc, etc., succeed each other in the order from *right to left*. Hence the two figures cannot be made to coincide by superposition, but are said to be *equal by symmetry*.

PROPOSITION XXIII. THEOREM.

487. *The sum of any two face angles of a trihedral angle is greater than the third.*



Let S–ABC be a trihedral angle in which the face angle ASC is greater than either angle ASB or angle BSC.

We are to prove $\angle ASB + \angle BSC > \angle ASC$.

In the face ASC draw SD, making $\angle ASD = \angle ASB$.

Through any point D of SD draw any straight line ADC cutting AS and SC.

Take $SB = SD$.

Pass a plane through AC and the point B.

In the $\triangle$s ASD and ASB

$AS = AS$, Iden.

$SD = SB$, Cons.

$\angle ASD = \angle ASB$. Cons.

$\therefore \triangle ASD = \triangle ASB$, § 106

$\therefore AD = AB$,

(being homologous sides of equal $\triangle$s*).*

In the $\triangle ABC$, $AB + BC > AC$.

Subtract the equals AB and AD.

Then $BC > DC$.

Now in the $\triangle$s BSC and DSC

$SB = SD$, Cons.

$SC = SC$, Iden.

but $BC > DC$,

$\therefore \angle BSC > \angle DSC$. § 116

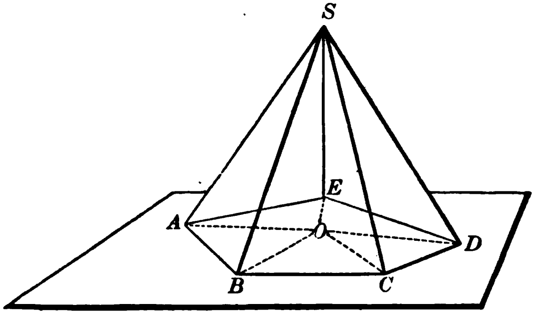
$\therefore \angle ASB + \angle BSC > \angle ASD + \angle DSC$,

that is $\angle ASB + \angle BSC > \angle ASC$.

Q. E. D

PROPOSITION XXIV. THEOREM.

488. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Let the polyhedral angle S be cut by a plane, making the section $ABCDE$ a convex polygon.

We are to prove $\angle ASB + \angle BSC$ *etc.* $<$ 4 *rt.* $\angle$s.

From any point O within the polygon draw OA, OB, OC, OD, OE.

The number of the $\triangle$s having their common vertex at O will be the same as the number having their common vertex at S.

$\therefore$ the sum of all the $\angle$s of the $\triangle$s having the common vertex at S is equal to the sum of all the $\angle$s of the $\triangle$s having the common vertex at O.

But in the trihedral $\angle$s formed at A, B, C, etc.

$$\angle SAE + \angle SAB > \angle OAE + \angle OAB, \qquad \S\ 487$$

(*the sum of any two face $\angle$s of a trihedral $\angle$ is greater than the third*).

and $$\angle SBA + \angle SBC > \angle OBA + \angle OBC. \qquad \S\ 487$$

$\therefore$ the sum of the $\angle$s at the bases of the $\triangle$s whose common vertex is S is greater than the sum of the $\angle$s at the bases of the $\triangle$s whose common vertex is O.

$\therefore$ the sum of the $\angle$s at S is less than the sum of the $\angle$s at O.

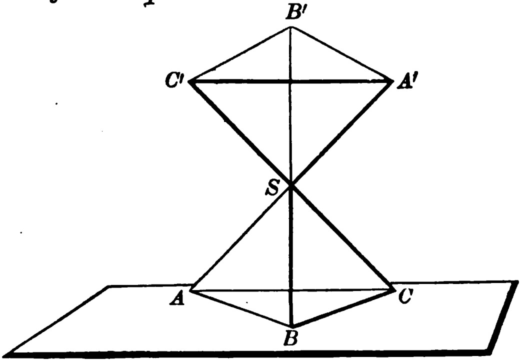
But the sum of the $\angle$s at O = 4 rt. $\angle$s. § 34

$\therefore$ the sum of the $\angle$s at S is less than 4 rt. $\angle$s.

Q. E. D.

PROPOSITION XXV. THEOREM.

489. *An isosceles trihedral angle and its symmetrical trihedral angle are equal.*



Let $S\text{-}ABC$ be an isosceles trihedral angle, having $\angle ASB = \angle BSC$. And let $S\text{-}A'B'C'$ be its symmetrical trihedral angle.

We are to prove trihedral $\angle S\text{-}ABC =$ *trihedral* $\angle S\text{-}A'B'C'$.

Revolve $\angle S\text{-}A'B'C'$ about S until SB' falls on SB and the plane $SB'A'$ falls on the plane SBC.

Now the dihedral $\angle C\text{-}SB\text{-}A$ = dihedral $\angle A'\text{-}SB'\text{-}C'$,

(*being vertical dihedral* $\measuredangle$s).

$\therefore$ the plane $SB'C'$ will fall on the plane SBA.

Now $\angle BSC = \angle BSA$, Hyp.

and $\angle B'SA' = \angle BSA$,

(*being vertical* $\measuredangle$s).

$\therefore \angle BSC = \angle B'SA'$; Ax. 1

$\therefore SA'$ will fall on SC.

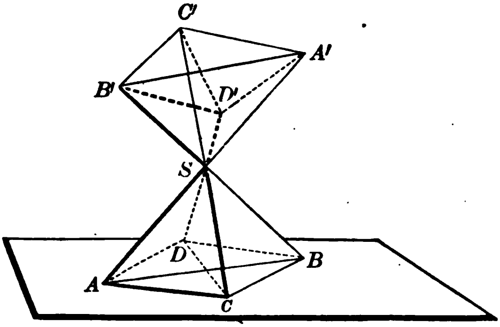
In like manner SC' will fall on SA,

$\therefore$ the two trihedral $\measuredangle$s will coincide and be equal.

Q. E. D.

PROPOSITION XXVI. THEOREM.

490. *Two symmetrical trihedral angles are equivalent.*



Let the trihedral $\angle S\text{-}ABC$ and $\angle S\text{-}A'B'C'$ be symmetrical.

We are to prove trihedral $\angle S\text{-}ABC \approxeq$[*] *trihedral* $\angle S\text{-}A'B'C'$.

Draw $D'D$ making the $\angle$s DSA, DSC, and DSB equal.

Then $\angle D'SA' = \angle D'SC' = \angle D'SB'$,

(being vertical $\angle$s *of the equal* $\angle$s DSA, DSC, *and* DSB).

Then the trihedral $\angle S\text{-}DCB =$ trihedral $\angle S\text{-}D'C'B'$, § 489

(two isosceles symmetrical trihedral $\angle$s *are equal).*

And trihedral $\angle S\text{-}DCA =$ trihedral $\angle S\text{-}D'C'A'$,

and trihedral $\angle S\text{-}ADB =$ trihedral $\angle S\text{-}A'D'B'$.

Adding the first two equalities, the polyhedral $\angle S\text{-}ABCD$ $\approxeq$ polyhedral $\angle S\text{-}A'B'C'D'$.

Take away from each of these equals the equal trihedral $\angle$s $S\text{-}ADB$ and $S\text{-}A'D'B'$.

Then trihedral $\angle S\text{-}ABC \approxeq$ trihedral $\angle S\text{-}A'B'C'$.

Q. E. D.

491. SCHOLIUM. If DD' fall within the given trihedral angles these trihedral angles would be composed of three isosceles trihedral angles which would be respectively equal, and hence the given trihedral angles would be equivalent.

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[*] The symbol ($\approxeq$) is to be read "equivalent to."

EXERCISES.

1. If a plane be passed through one of the diagonals of a parallelogram, the perpendiculars to the plane from the extremities of the other diagonal are equal.

2. If each of the projections of a line *A B* upon two intersecting planes be a straight line, the line *A B* is a straight line.

3. The height of a room is eight feet, how can a point in the floor directly under a certain point in the ceiling be determined with a ten-foot pole?

4. If a line be drawn at an inclination of 45° to a plane, what is the greatest angle which any line of the plane, drawn through the point in which the inclined line pierces the plane, makes with the line.

5. Through a given point pass a plane parallel to a given plane.

6. Find the locus of points in space which are equally distant from two given points.

7. Show that the three planes embracing the edges of a trihedral angle and the bisectors of the opposite face angles respectively intersect in the same straight line.

8. Find the locus of the points which are equally distant from the three edges of a trihedral angle.

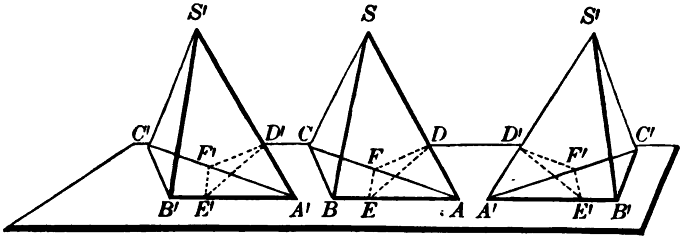
9. Cut a given quadrahedral angle by a plane so that the section shall be a parallelogram.

10. Determine a point in a given plane such that the sum of its distances from two given points on the same side of the plane shall be a minimum.

11. Determine a point in a given plane such that the difference of its distances from two given points on opposite sides of a plane shall be a maximum.

PROPOSITION XXVII. THEOREM.

492. *Two trihedral angles are equal or symmetrical when the three face angles of the one are respectively equal to the three face angles of the other.*



In the trihedral $\measuredangle$ S and S', let $\angle ASB = \angle A'S'B'$, $\angle ASC = \angle A'S'C'$, and $\angle BSC = \angle B'S'C'$.

We are to prove that the homologous dihedral angles are equal, and hence the trihedral angles S and S' are either equal or symmetrical.

On the edges of these angles take the six equal distances SA, SB, SC, $S'A'$, $S'B'$, $S'C'$.

Draw AB, BC, AC, $A'B'$, $B'C'$, $A'C'$.

The homologous isosceles $\triangle$ SAB, $S'A'B'$, SAC, $S'A'C'$, SBC, $S'B'C'$ are equal, respectively. § 106

$\therefore AB$, AC, BC equal respectively $A'B'$, $A'C'$, $B'C'$,
(being homologous sides of equal $\triangle$).

$\therefore \triangle ABC = \triangle A'B'C'$. § 108

At any point D in SA draw DE and $DF \perp$ to SA in the faces ASB and ASC respectively.

These lines meet AB and AC respectively,
(since the $\measuredangle$ SAB and SAC are acute, each being one of the equal $\measuredangle$ of an isosceles $\triangle$).

Join EF.

On $S'A'$ take $A'D' = AD$.

Draw $D'E'$ and $D'F'$ in the faces $A'S'B'$ and $A'S'C'$ respectively $\perp$ to $S'A'$, and join $E'F'$.

In the rt. $\triangle$s ADE and $A'D'E'$

$$AD = A'D', \qquad \text{Cons.}$$

$$\angle DAE = \angle D'A'E',$$

(*being homologous* $\angle$s *of the equal* $\triangle$s SAB *and* $S'A'B'$).

$$\therefore \text{rt. } \triangle ADE = \text{rt. } \triangle A'D'E', \qquad \S\ 111$$

$$\therefore AE = A'E' \text{ and } DE = D'E',$$

(*being homologous sides of equal* $\triangle$s).

In like manner we may prove $AF = A'F'$ and $DF = D'F'$.

Hence in the $\triangle$s AEF and $A'E'F'$ we have

$$AE \text{ and } AF = \text{respectively } A'E' \text{ and } A'F',$$

and
$$\angle EAF = \angle E'A'F',$$

(*being homologous* $\angle$s *of the equal* $\triangle$s ABC *and* $A'B'C'$).

$$\therefore \triangle AEF = \triangle A'E'F', \qquad \S\ 106$$

$$\therefore EF = E'F'$$

(*being homologous sides of the equal* $\triangle$s AEF *and* $A'E'F'$).

Hence, in the $\triangle$s EDF and $E'D'F'$ we have

$$ED,\ DF, \text{ and } EF = \text{respectively } E'D',\ D'F', \text{ and } E'F'.$$

$$\therefore \triangle EDF = \triangle E'D'F', \qquad \S\ 108$$

$$\therefore \angle EDF = \angle E'D'F',$$

(*being homologous* $\angle$s *of equal* $\triangle$s).

$$\therefore \text{the dihedral } \angle\ B\text{-}AS\text{-}C = \text{dihedral } \angle\ B'\text{-}A'S'\text{-}C',$$

(*since* $\angle$s EDF *and* $E'D'F'$, *the measures of these dihedral* $\angle$s, *are equal*).

In like manner it may be proved that the dihedral $\angle$s $A\text{-}BS\text{-}C$ and $A\text{-}CS\text{-}B$ are equal respectively to the dihedral $\angle$s $A'\text{-}B'S'\text{-}C'$ and $A'\text{-}C'S'\text{-}B'$.

Q. E. D.

This demonstration applies to either of the two figures denoted by $S'\text{-}A'B'C'$, which are symmetrical with respect to each other. If the first of these figures be given, S and S' are equal, for they can be applied to each other so as to coincide in all their parts. If the second be given, S and S' are symmetrical. § 486

BOOK VII.

POLYHEDRONS, CYLINDERS, AND CONES.

GENERAL DEFINITIONS.

493. DEF. A *Polyhedron* is a solid bounded by four or more polygons.

A polyhedron bounded by four polygons is called a *tetrahedron;* by six, a *hexahedron;* by eight, an *octahedron;* by twelve, a *dodecahedron;* by twenty, an *icosahedron.*

494. DEF. The *Faces* of a polyhedron are the bounding polygons.

495. DEF. The *Edges* of a polyhedron are the intersections of its faces.

496. DEF. The *Vertices* of a polyhedron are the intersections of its edges.

497. DEF. A *Diagonal* of a polyhedron is a straight line joining any two vertices not in the same face.

498. DEF. A *Section* of a polyhedron is a polygon formed by the intersection of a plane with three or more faces.

499. DEF. A *Convex* polyhedron is a polyhedron every section of which is a convex polygon.

500. DEF. The *Volume* of a polyhedron is the numerical measure of its magnitude referred to some other polyhedron as a unit of measure.

501. DEF. The polyhedron adopted as the unit of measure is called the *Unit of Volume.*

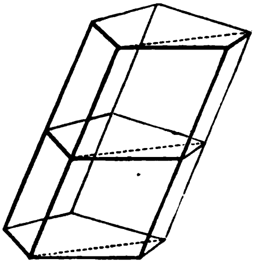
502. DEF. *Similar* polyhedrons are polyhedrons which have the same *form.*

503. DEF. *Equivalent* polyhedrons are polyhedrons which have the same *volume.*

504. DEF. *Equal* polyhedrons are polyhedrons which have the same *form* and *volume.*

ON PRISMS.

505. DEF. A *Prism* is a polyhedron two of whose faces are equal and parallel polygons, and the other faces are parallelograms.



OBLIQUE PRISM.

506. Def. The *Bases* of a prism are the equal and parallel polygons.

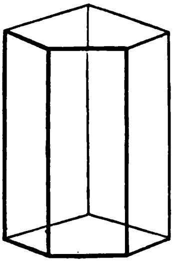
507. Def. The *Lateral* faces of a prism are all the faces except the bases.

508. Def. The *Lateral* or *Convex Surface* of a prism is the sum of its lateral faces.

509. Def. The *Lateral edges* of a prism are the intersections of its lateral faces; the *Basal edges* of a prism are the intersections of the bases with the lateral faces.

510. Def. Prisms are *triangular*, *quadrangular*, *pentagonal*, etc., according as their bases are triangles, quadrangles, pentagons, etc.

511. Def. A *Right* prism is a prism whose lateral edges are perpendicular to its bases.



RIGHT PRISM.

512. Def. An *Oblique* prism is a prism whose lateral edges are oblique to its bases.

513. Def. A *Regular* prism is a right prism whose bases are regular polygons, and hence its lateral faces are equal rectangles.

514. Def. The *Altitude* of a prism is the perpendicular distance between the planes of its bases. The altitude of a right prism is equal to any one of its lateral edges.

515. Def. A *Truncated* prism is a portion of a prism included between either base and a section inclined to the base and cutting all the lateral edges.

516. Def. A *Right* section of a prism is a section perpendicular to its lateral edges.

517. Def. A *Parallelopiped* is a prism whose bases are parallelograms.

518. Def. A *Right* parallelopiped is a parallelopiped whose lateral edges are perpendicular to its bases; hence its lateral faces are rectangles.

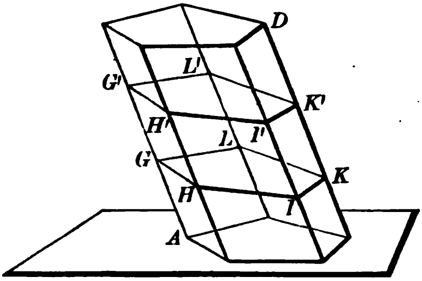
519. Def. An *Oblique* parallelopiped is a parallelopiped whose lateral edges are oblique to its bases.

520. Def. A *Rectangular* parallelopiped is a right parallelopiped whose bases are rectangles.

521. Def. A *Cube* is a rectangular parallelopiped all of whose faces are squares.

PROPOSITION I. THEOREM.

522. *The sections of a prism made by parallel planes are equal polygons.*



Let the prism AD be intersected by the parallel planes GK, $G'K'$.

We are to prove section $GHIKL$ = section $G'H'I'K'L'$.

GH, HI, IK, etc., are parallel respectively to $G'H'$, $H'I'$, $I'K'$, etc., § 465

(*the intersections of two* ∥ *planes by a third plane are* ∥ *lines*).

∴ ⦞ GHI, HIK, etc., are equal respectively to ⦞ $G'H'I'$, $H'I'K'$, etc., § 462

(*two* ⦞ *not in the same plane, having their sides respectively parallel and lying in the same direction, are equal*).

Also, sides GH, HI, IK, etc., are equal respectively to $G'H'$, $H'I'$, $I'K'$, etc., § 135

(∥ *lines comprehended between* ∥ *lines are equal*).

∴ section $GHIKL$ = section $G'H'I'K'L'$, § 155

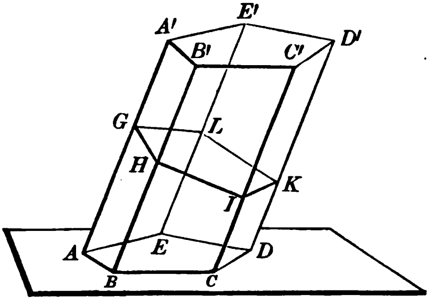
(*being mutually equiangular and equilateral*).

Q. E. D.

523. COROLLARY. Any section of a prism parallel to the base is equal to the base; and all right sections of a prism are equal.

PROPOSITION II. THEOREM.

524. *The lateral area of a prism is equal to the product of a lateral edge by the perimeter of the right section.*



Let $GHIKL$ be a right section of the prism AD'.

We are to prove lateral area of prism $AD' = AA' \times$ perimeter $GHIKL$.

Consider the lateral edges AA', BB', etc., to be the bases of the ▱s AB', BC', etc., which form the convex surface of the prism.

Then the altitudes of these ▱s will be the ⊥s GH, HI, IK, etc.,

and the area of each ▱ is the product of its base and altitude. § 321

Now the bases of these ▱s are all equal, § 464

(∥ *lines comprehended between* ∥ *planes are equal*);

and the sum of the altitudes GH, HI, IK, etc., is the perimeter of the right section.

Hence, the sum of the areas of these ▱s is the product of a lateral edge AA' by the perimeter of the right section.

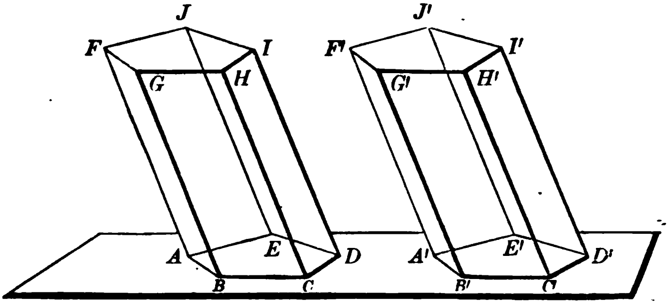
That is, the lateral area of the prism is equal to the product of a lateral edge by the perimeter of a right section.

Q. E. D.

525. COROLLARY. The lateral area of a right prism is equal to the altitude multiplied by the perimeter of the base.

PROPOSITION III. THEOREM.

526. *Two prisms are equal if the three faces including a trihedral angle of the one be respectively equal to the three corresponding faces including a trihedral angle of the other, and similarly placed.*



Let AD, AG, AJ, be respectively equal to $A'D'$, $A'G'$, $A'J'$, and similarly placed.

We are to prove prism AI = prism $A'I'$.

Now trihedral $\angle A$ = trihedral $\angle A'$, § 492

(two trihedrals are equal, when the three face $\angle$s of the one are equal respectively to the three face $\angle$s of the other and are similarly placed).

Apply trihedral $\angle A$ to trihedral $\angle A'$.

Then the base AD will coincide with the base $A'D'$,

face AG with $A'G'$,

and face AJ with $A'J'$;

$\therefore$ FG will coincide with $F'G'$, and FJ with $F'J'$.

$\therefore$ the upper bases, FI and $F'I'$, will coincide,

(being equal polygons, since they are equal to the equal lower bases).

$\therefore$ the remaining edges will coincide,

(their extremities being the same points).

$\therefore$ the prisms will coincide and be equal.

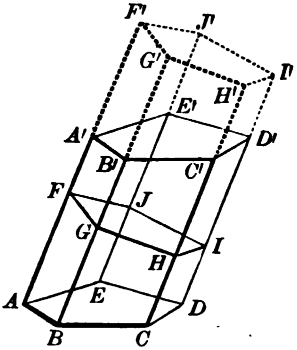
Q. E. D.

527. COROLLARY 1. Two truncated prisms are equal, if the three faces including a trihedral of the one be respectively equal to the three faces including a trihedral of the other, and be similarly placed.

528. COR. 2. Two right prisms having equal bases and altitudes are equal. If the faces be not similarly placed, if one be inverted, the faces will be similarly placed and the prisms can be made to coincide.

PROPOSITION IV. THEOREM.

529. *An oblique prism is equivalent to a right prism whose bases are equal to right sections of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Let AD' be an oblique prism, and FI a right section.

Complete the right prism FI', making its edges equal to those of the oblique prism.

We are to prove oblique prism $AD' \Leftrightarrow$ right prism FI'.

In the prisms AI and $A'I'$

trihedral $\angle A =$ trihedral $\angle A'$, § 492

(two trihedrals are equal when three face $\angle s$ of the one are respectively equal to three face $\angle s$ of the other, and are similarly placed).

Now face $AD =$ face $A'D'$, § 505

(being the two bases of the oblique prism AD');

face $AJ =$ face $A'J'$, Cons.

and face $AG =$ face $A'G'$. Cons.

$\therefore$ prism $AI =$ prism $A'I'$, § 527

(two truncated prisms are equal when the three faces including a trihedral of the one are respectively equal to the three faces including a trihedral of the other, and are similarly placed).

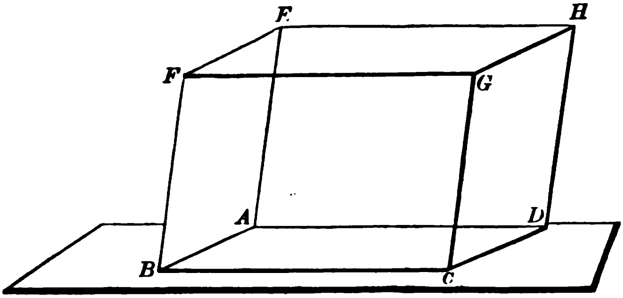
To each of these equal prisms add the prism FD'.

Then oblique prism $AD' \Leftrightarrow$ right prism FI'.

Q. E. D.

PROPOSITION V. THEOREM.

530. *Any two opposite faces of a parallelopiped are equal and parallel.*



Let $A G$ be a parallelopiped.

We are to prove faces $A F$ and $D G$ equal and parallel.

Since $A C$ is a ▱, § 517

$A B$ and $D C$ are equal and ∥ lines. § 125

Also, since $A H$ is a ▱, § 505

$A E$ and $D H$ are equal and ∥ lines. § 125

$\therefore \angle E A B = \angle H D C$, § 462

(*two* ∆ *not in the same plane having their sides* ∥ *and lying in the same direction are equal*).

$\therefore$ face $A F =$ face $D G$. § 140

Moreover, face $A F$ is ∥ to $D G$, § 463

(*if two* ∆ *not in the same plane have their sides* ∥ *and lying in the same direction their planes are parallel*).

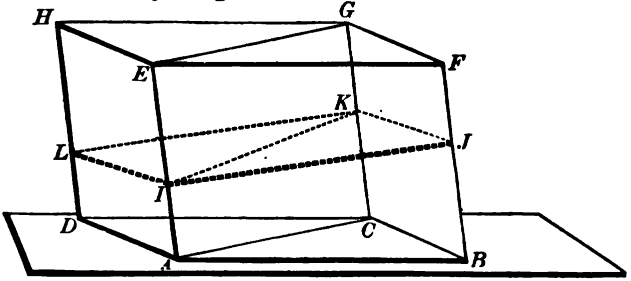
In like manner we may prove $A H$ and $B G$ equal and parallel.

Q. E. D.

531. SCHOLIUM. Any two opposite faces of a parallelopiped may be taken for bases, since they are equal and parallel parallelograms.

PROPOSITION VI. THEOREM.

532. *The plane passed through two diagonally opposite edges of a parallelopiped divides the parallelopiped into two equivalent triangular prisms.*



Let the plane $AEGC$ pass through the opposite edges AE and CG of the parallelopiped AG.

We are to prove that the parallelopiped AG is divided into two equivalent triangular prisms, ABC-F, and ADC-H.

Let $IJKL$ be a right section of the parallelopiped made by a plane ⊥ to the edge AE.

The intersection IK of this plane with the plane $AEGC$ is the diagonal of the ▱ $IJKL$.

$\therefore \triangle IKJ = \triangle IKL$. § 133

But prism ABC-F is equivalent to a right prism whose base is IJK and whose altitude is AE, § 529

(*any oblique prism is* ⇔ *to a right prism whose bases are equal to right sections of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism*).

The prism ADC-H is equivalent to a right prism whose base is ILK, and whose altitude is AE. § 529

Now the two right prisms are equal, § 528

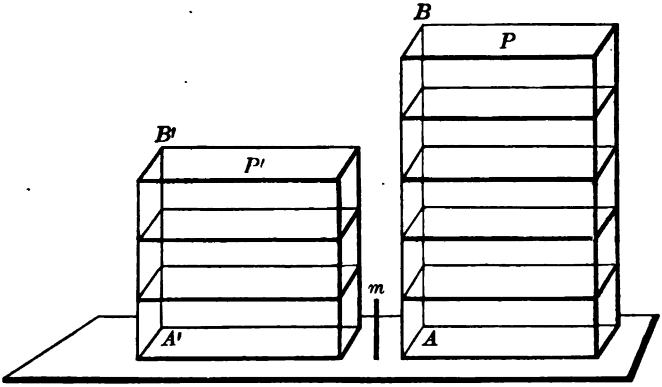
(*two right prisms having equal bases and altitudes are equal*).

$\therefore ABC$-F ⇔ ADC-H.

Q. E. D.

Proposition VII. Theorem.

533. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*



Let AB and $A'B'$ be the altitudes of the two rectangular parallelopipeds, P, and P', having equal bases.

We are to prove $$\frac{P}{P'} = \frac{AB}{A'B'}.$$

Case I. — *When AB and $A'B'$ are commensurable.*

Find a common measure m, of AB and $A'B'$.

Suppose m to be contained in AB 5 times, and in $A'B'$ 3 times.

Then we have $$\frac{AB}{A'B'} = \frac{5}{3}.$$

At the several points of division on AB and $A'B'$ pass planes ⊥ to these lines.

The parallelopiped P will be divided into 5,

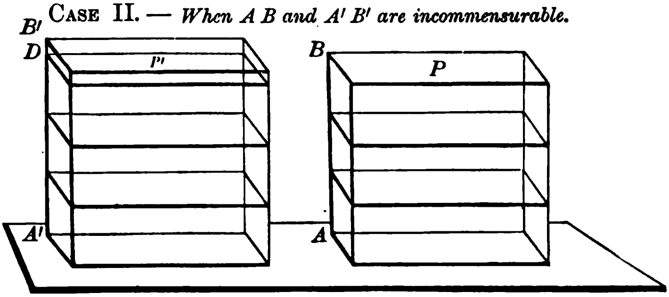
and P' into 3, parallelopipeds equal, each to each, § 528

(two right prisms having equal bases and altitudes are equal).

Then $$\frac{P}{P'} = \frac{5}{3}.$$

$$\therefore \frac{P}{P'} = \frac{AB}{A'B'}.$$

CASE II. — *When A B and A′ B′ are incommensurable.*



Let $A\,B$ be divided into any number of equal parts,

and let one of these parts be applied to $A'\,B'$ as many times as $A'\,B'$ will contain it.

Since $A\,B$ and $A'\,B'$ are incommensurable, a certain number of these parts will extend from A' to a point D, leaving a remainder $D\,B'$ less than one of these parts.

Through D pass a plane $\perp$ to $A'\,B'$, and denote the parallelopiped whose base is the same as that of P', and whose altitude is $A'\,D$ by Q.

Now, since $A\,B$ and $A'\,D$ are commensurable,

$$Q : P = A'\,D : A\,B. \qquad \text{(Case I.)}$$

Suppose the number of parts into which $A\,B$ is divided to be continually increased, the length of each part will become less and less, and the point D will approach nearer and nearer to B'.

The limit of Q will be P',

and the limit of $A'\,D$ will be $A'\,B'$,

$\therefore$ the limit of $Q : P$ will be $P' : P$,

and the limit of $A'\,D : A\,B$ will be $A'\,B' : A\,B$,

Moreover the corresponding values of the two variables $Q : P$ and $A'\,D : A\,B$ are always equal, however near these variables approach their limits.

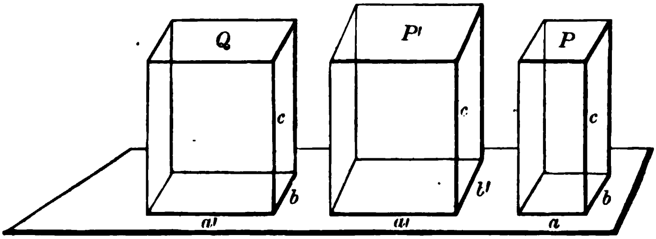
$\therefore$ their limits $P' : P = A'\,B' : A\,B$. § 199

Q. E. D.

534. SCHOLIUM. The three edges of a rectangular parallelopiped which meet at a common vertex are its dimensions. Hence two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions.

PROPOSITION VIII. THEOREM.

535. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Let a, b, **and** c, **and** a', b', c, **be the three dimensions respectively of the two rectangular parallelopipeds** P **and** P'.

We are to prove $\frac{P}{P'} = \frac{a \times b}{a' \times b'}$.

Let Q be a third rectangular parallelopiped whose dimensions are a', b and c.

Now Q has the two dimensions b and c in common with P, and the two dimensions a' and c in common with P'.

Then $\frac{P}{Q} = \frac{a}{a'}$, § 534

(*two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions*);

and $\frac{Q}{P'} = \frac{b}{b'}$. § 534

Multiply these two equalities together;

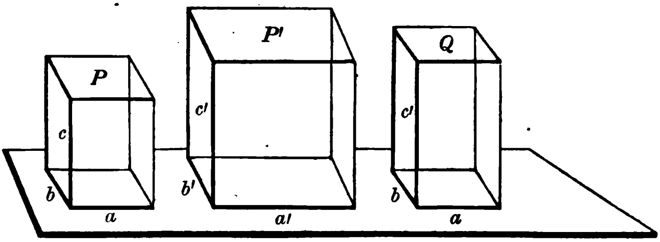
then $\frac{P}{P'} = \frac{a \times b}{a' \times b'}$.

Q. E. D.

536. SCHOLIUM. This proposition may be stated thus: two rectangular parallelopipeds which have one dimension in common are to each other as the products of the other two dimensions.

PROPOSITION IX. THEOREM.

537. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Let a, b, c, and a', b', c', be the three dimensions respectively of the two rectangular parallelopipeds P and P'.

We are to prove $\dfrac{P}{P'} = \dfrac{a \times b \times c}{a' \times b' \times c'}$.

Let Q be a third rectangular parallelopiped whose dimensions are a, b, and c'.

Then $$\frac{P}{Q} = \frac{c}{c'}, \qquad \S\ 534$$

(two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions);

and $$\frac{Q}{P'} = \frac{a \times b}{a' \times b'}, \qquad \S\ 536$$

(two rectangular parallelopipeds which have one dimension in common are to each other as the products of their other two dimensions).

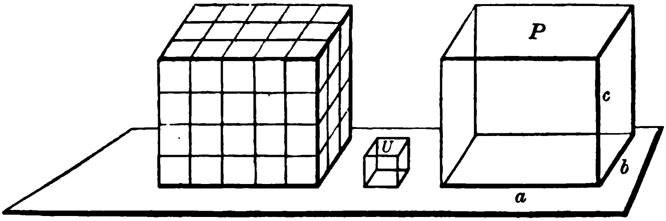
Multiply these equalities together;

then $$\frac{P}{P'} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

Q. E. D.

PROPOSITION X. THEOREM.

538. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions, the unit of volume being a cube whose edge is the linear unit.*



Let a, b, and c be the three dimensions of the rectangular parallelopiped P, and let the cube U be the unit of volume.

We are to prove *volume of* $P = a \times b \times c$.

$$\frac{P}{U} = \frac{a \times b \times c}{1 \times 1 \times 1}. \qquad \text{§ 537}$$

But $\frac{P}{U}$ is the volume of P; § 500

$\therefore$ the volume of $P = a \times b \times c$.

Q. E. D.

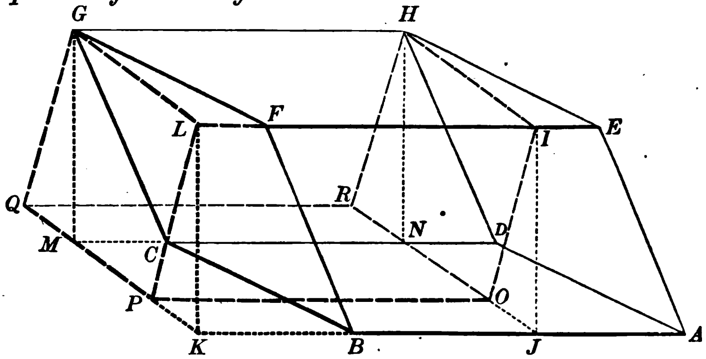
539. COROLLARY I. Since a cube is a rectangular parallelopiped having its three dimensions equal, the *volume of a cube is equal to the third power of its edge.*

540. COR. II. The product $a \times b$ represents the base when c is the altitude; hence: *The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.*

541. SCHOLIUM. When the three dimensions of the rectangular parallelopiped are each exactly divisible by the linear unit, this proposition is rendered evident by dividing the solid into cubes, each equal to the unit of volume. Thus, if the three edges which meet at a common vertex contain the linear unit 3, 4 and 5 times respectively, planes passed through the several points of division of the edges, and perpendicular to them, will divide the solid into cubes, each equal to the unit of volume; and there will evidently be 3 × 4 × 5 of these cubes.

Proposition XI. Theorem.

542. *The volume of any parallelopiped is equal to the product of its base by its altitude.*



Let *ABCD-F* be a parallelopiped having all its faces oblique, and *HR* its altitude.

We are to prove $ABCD\text{-}F = ABCD \times HR$.

By making the right section $HIJN$ and completing the parallelopiped $HIJN\text{-}GLKM$ we have a right parallelopiped equivalent to, $ABCD\text{-}F$. § 529

(*an oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism*).

Through the edge IL make the right section $ILPO$, and complete the right parallelopiped $ILPO\text{-}HGQR$, and we have a rectangular parallelopiped equivalent to $HIJN\text{-}GLKM$, § 529

and hence equivalent to $ABCD\text{-}F$.

Now $\square\, ILGH \Bumpeq \square\, EFGH$, § 322

$\square\, OPQR = (\square\, ILGH) = \square\, JKMN$; § 530

and $\square\, ABCD = \square\, EFGH$. § 530

$\therefore \square\, OPQR \Bumpeq \square\, ABCD$.

Moreover, the three parallelopipeds have the common altitude HR.

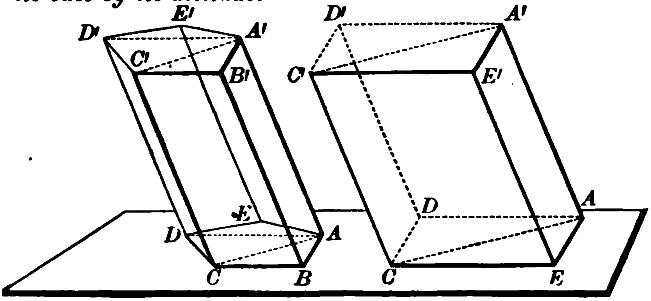
But $OPQR\text{-}ILGH = OPQR \times HR$; § 540

$\therefore ABCD\text{-}F = ABCD \times HR$.

Q. E. D.

PROPOSITION XII. THEOREM.

543. *The volume of any prism is equal to the product of its base by its altitude.*



CASE I. — *When the base is a triangle.*

Let V denote the volume, B the base, and H the altitude of the triangular prism $A\,E\,C\text{-}E'$.

We are to prove $V = B \times H$.

Upon the edges $A\,E$, $E\,C$, $E\,E'$, construct parallelopiped $A\,E\,C\,D\text{-}E'$.

Then $A\,E\,C\text{-}E' \Leftrightarrow \frac{1}{2}\,A\,E\,C\,D\text{-}E'$, § 532

(*the plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms*),

and $A\,E\,C = \frac{1}{2}\,A\,E\,C\,D$. § 133

But $A\,E\,C\,D\text{-}E' = 2\,B \times H$, § 542

(*the volume of any parallelopiped is equal to the product of its base by its altitude*).

$$\therefore V = \tfrac{1}{2}(2\,B \times H) = B \times H.$$

CASE II. — *When the base is a polygon of more than three sides.*

Planes passed through the lateral edge $A\,A'$, and the several diagonals of the base will divide the given prism into triangular prisms,

which have for a common altitude the altitude of the prism.

Hence, the volume of the entire prism is the product of the sum of their bases by the common altitude,

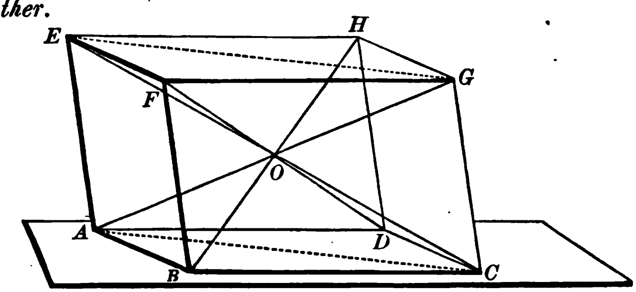
that is the entire base by the altitude of the prism.

Q. E. D.

544. COROLLARY. Prisms having equivalent bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases; and any two prisms are to each other as the product of their bases and altitudes. Any two prisms havi[illegible] [illegible]nt bases and equal altitudes are equivalent.

PROPOSITION XIII. THEOREM.

545. *The four diagonals of a parallelopiped bisect each other.*



Let A G, E C, B H, and F D, be the four diagonals of the parallelopiped A G.

We are to prove these four diagonals bisect each other.

Through the opposite and ‖ edges *A E* and *C G* pass a plane intersecting the ‖ bases in the ‖ lines *A C* and *E G*.

The section *A C G E* is a ▱,
(having its opposite sides ‖);

∴ its diagonals *A G* and *E C* bisect each other in the point *O*. § 138

In like manner a plane passed through the opposite and ‖ edges *F G* and *A D* will form a ▱ *A F G D*,

whose diagonals *A G* and *F D* will bisect each other in the point *O*. § 138

Also, a plane passed through the opposite and ‖ edges *E H* and *B C* will form a ▱ *E B C H*,

whose diagonals *E C* and *B H* will bisect each other in the point *O*.

∴ the four diagonals bisect each other at the point *O*.

Q. E. D.

546. COROLLARY. The diagonals of a rectangular parallelopiped are equal.

547. SCHOLIUM. The point *O*, in which the four diagonals intersect, is called the centre of the parallelopiped; and it is evident that any straight line drawn through the point *O* and terminated by two opposite faces of the parallelopiped is bisected at that point. Hence *O* is the centre of symmetry.

ON PYRAMIDS.

548. DEF. A *Pyramid* is a polyhedron one of whose faces is a polygon, and whose other faces are triangles having a common vertex and the sides of the polygon for bases.

549. DEF. The *Base* of a pyramid is the face whose sides are the bases of the triangles having a common vertex.

550. DEF. The *Lateral faces* of a pyramid are all the faces except the base.

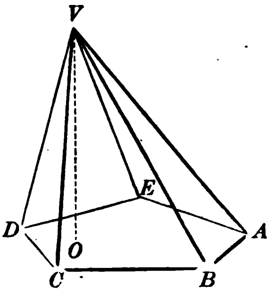
551. DEF. The *Lateral surface* of a pyramid is the sum of its lateral faces.

552. DEF. The *Lateral edges* of a pyramid are the intersections of its lateral faces.

553. DEF. The *Basal edges* of a pyramid are the intersections of its base with its lateral faces.

554. DEF. The *Vertex* of a pyramid is the common vertex of its lateral faces.

555. DEF. The *Altitude* of a pyramid is the perpendicular distance from its vertex to the plane of its base.



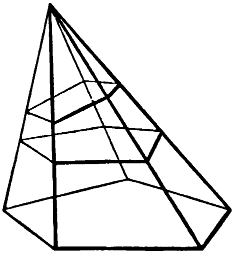
Thus, *V-A B C D E* is a pyramid; *A B C D E* is its base; *A V B*, *B V C*, etc. are its lateral faces, and their sum is its lateral surface; *V A*, *V B*, etc. are its lateral edges; *A B*, *B C*, etc. its basal edges; *V* is its vertex; *V O* is its altitude.

556. DEF. A *Regular* pyramid is a pyramid whose base is a regular polygon, and whose vertex is in the perpendicular to the base at its centre.

557. DEF. The *Axis* of a regular pyramid is the straight line joining its vertex with the centre of the base.

558. DEF. The *Slant height* of a regular pyramid is the altitude of any lateral face.

559. DEF. A pyramid is *triangular*, *quadrangular*, *pentagonal*, etc., according as its base is a triangle, quadrilateral, pentagon, etc. A triangular pyramid formed by four faces (all of which are triangles) is a *tetrahedron*.



560. DEF. A *Truncated* pyramid is the portion of a pyramid included between its base and a section cutting all its lateral edges.

561. DEF. A *Frustum* of a pyramid is a truncated pyramid in which the cutting section is parallel to the base.

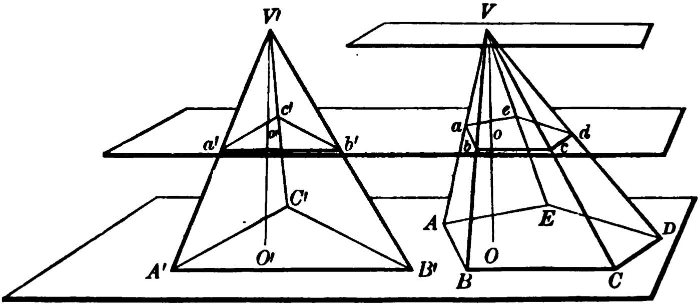
562. DEF. The base of the pyramid is called the *Lower base* of the frustum, and the parallel section, its *Upper base.*

563. DEF. The *Altitude* of a frustum is the perpendicular distance between the planes of its bases.

564. DEF. The lateral faces of a frustum of a regular pyramid are the trapezoids included between its bases; the lateral surface is the sum of the lateral faces; the *Slant height* of a frustum of a regular pyramid is the altitude of any lateral face.

PROPOSITION XIV. THEOREM.

565. *If a pyramid be cut by a plane parallel to its base,*
I. *The edges and altitude are divided proportionally;*
II. *The section is a polygon similar to the base.*



Let the pyramid *V-A B C D E*, **whose altitude is** *V O*, **be cut by a plane** *a b c d e* **parallel to its base, intersecting the lateral edges in the points** *a*, *b*, *c*, *d*, *e*, **and the altitude in** *o*.

We are to prove

I. $$\frac{Va}{VA} = \frac{Vb}{VB} \cdots\cdot = \frac{Vo}{VO};$$

II. *The section a b c d e similar to the base A B C D E.*

I. Suppose a plane passed through the vertex *V* ∥ to the base.

Then the edges and the altitude will be intersected by three ∥ planes.

$$\therefore \frac{Va}{VA} = \frac{Vb}{VB} \cdots\cdot = \frac{Vo}{VO}, \qquad \S\ 469$$

(*if straight lines be intersected by three ∥ planes, their corresponding segments are proportional*).

II. The sides *a b*, *b c* etc. are parallel respectively to *A B*, *B C*, etc., § 465

(*the intersections of ∥ planes by a third plane are ∥ lines*);

∴ ∠s *a b c*, *b c d* etc. are equal respectively to ∠s *A B C*, *B C D* etc., § 462

(*if two ∠s not in the same plane have their sides respectively ∥ and lying in the same direction, they are equal*).

∴ the two polygons are mutually equiangular.

Also, since the sides of the section are ∥ to the corresponding sides of the base,

$\triangle$s Vab, Vbc etc. are similar respectively to $\triangle$s VAB, VBC etc.

$$\therefore \frac{ab}{AB} = \left(\frac{Vb}{VB}\right) = \frac{bc}{BC} = \left(\frac{Vc}{VC}\right) = \frac{cd}{CD} \text{ etc.}$$

∴ the polygons have their homologous sides proportional;

∴ section $abcde$ is similar to the base $ABCDE$. § 278

Q. E. D.

566. Corollary 1. Any section of a pyramid, parallel to its base is to the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.

Since $$\frac{Vo}{VO} = \left(\frac{Vb}{VB}\right) = \frac{ab}{AB}.$$

Squaring $$\frac{\overline{Vo}^2}{\overline{VO}^2} = \frac{\overline{ab}^2}{\overline{AB}^2}.$$

But $$\frac{abcde}{ABCDE} = \frac{\overline{ab}^2}{\overline{AB}^2},$$ § 344

(similar polygons are to each other as the squares of their homologous sides).

$$\therefore \frac{abcde}{ABCDE} = \frac{\overline{Vo}^2}{\overline{VO}^2}.$$

567. Cor. 2. If two pyramids having equal altitudes be cut by planes parallel to their bases, and at equal distances from their vertices, the sections will have the same ratio as their bases.

For $$\frac{abcde}{ABCDE} = \frac{\overline{Vo}^2}{\overline{VO}^2},$$

and $$\frac{a'b'c'}{A'B'C'} = \frac{\overline{V'o'}^2}{\overline{V'O'}^2}.$$

Now, since $Vo = V'o'$, and $VO = V'O'$,

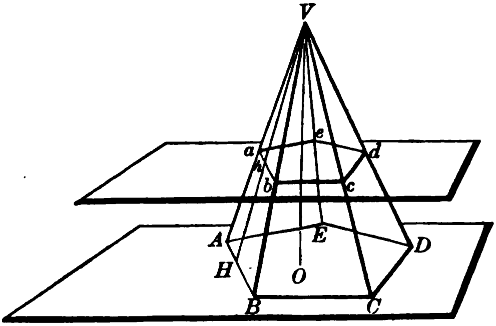
$$abcde : ABCDE :: a'b'c' : A'B'C'.$$

Whence $abcde : a'b'c' :: ABCDE : A'B'C'$. § 262

568. Cor. 3. If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to their bases and at equal distances from their vertices are equivalent.

PROPOSITION XV. THEOREM.

569. *The lateral area of a regular pyramid is equal to one-half the product of the perimeter of its base by its slant height.*



Let V-$ABCDE$ be a regular pyramid, and VH its slant height.

We are to prove the sum of the faces VAB, VBC, etc. $= \frac{1}{2}$ $(AB + BC$, etc.$) \times VH$.

Now $AB = BC = CD$, etc., § 363

(being sides of a regular polygon).

$VA = VB = VC$, etc., § 450

(oblique lines drawn from any point in a ⊥ to a plane at equal distances from the foot of the ⊥ are equal).

∴ △ VAB, VBC, etc. are equal isosceles △, § 108

whose bases are the sides of the regular polygon and whose common altitude is the slant height VH.

Now the area of one of these △, as VAB, $= \frac{1}{2}$ base $AB \times$ altitude VH, § 324

∴ the sum of the areas of these △, that is, the lateral area of the pyramid, is equal to $\frac{1}{2}$ the sum of their bases

$$(AB + BC + CD, \text{etc.}) \times VH.$$

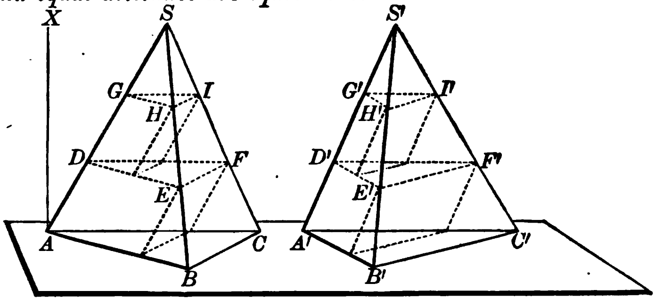
Q. E. D.

570. COROLLARY 1. *The lateral area of the frustum of a regular pyramid, being composed of trapezoids which have for their common altitude the slant height of the frustum, is equal to one-half the sum of the perimeters of the bases multiplied by the slant height of the frustum.*

571. COR. 2. The dihedral angles formed by the intersections of the lateral faces of a regular pyramid are all equal. § 492

PROPOSITION XVI. THEOREM.

572. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Let S-$A\,B\,C$ and S'-$A'\,B'\,C'$ be two triangular pyramids having equivalent bases $A\,B\,C$ and $A'\,B'\,C'$ situated in the same plane, and a common altitude $A\,X$.

We are to prove S-$A\,B\,C \Leftrightarrow S'$-$A'\,B'\,C'$.

Divide the altitude $A\,X$ into a number of equal parts,

and through the points of division pass planes ‖ to the planes of their bases, intersecting the two pyramids.

In the pyramids S-$A\,B\,C$ and S'-$A'\,B'\,C'$ inscribe prisms whose upper bases are the sections $D\,E\,F$, $G\,H\,I$, etc., $D'\,E'\,F'$, $G'\,H'\,I'$, etc.

The corresponding sections are equivalent, § 568
(if two pyramids have equal altitudes and equivalent bases, sections made by planes ‖ to their bases and at equal distances from their vertices are equivalent).

∴ the corresponding prisms are equivalent, § 544
(prisms having equivalent bases and equal altitudes are equivalent).

Denote the sum of the prisms inscribed in the pyramid S-$A\,B\,C$, and the sum of the corresponding prisms inscribed in the pyramid S'-$A'\,B'\,C'$ by V and V' respectively.

Then $V = V'$.

Now let the number of equal parts into which the altitude $A\,X$ is divided be indefinitely increased;

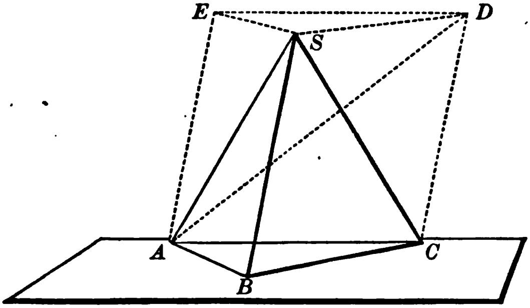
The volumes V and V' are always equal, and approach to the pyramids S-$A\,B\,C$ and S'-$A'\,B'\,C'$ respectively as their limits.

Hence S-$A\,B\,C \Leftrightarrow S'$-$A'\,B'\,C'$. § 199

Q. E. D.

PROPOSITION XVII. THEOREM.

573. *The volume of a triangular pyramid is equal to one-third of the product of its base and altitude.*



Let S-ABC be a triangular pyramid, and H its altitude.

We are to prove $S\text{-}ABC = \frac{1}{3} ABC \times H$.

On the base ABC construct a prism ABC-SED, having its lateral edges ∥ to SB and its altitude equal to that of the pyramid.

The prism will be composed of the triangular pyramid S-ABC and the quadrangular pyramid S-$ACDE$.

Through SA and SD pass a plane SAD.

This plane divides the quadrangular pyramid into the two triangular pyramids, S-ACD and S-AED, which have the same altitude and equal bases. § 133

$\therefore S\text{-}ACD \Leftrightarrow S\text{-}AED$, § 572

(*two triangular pyramids having equivalent bases and equal altitudes are equivalent*).

Now the pyramid S-AED may be regarded as having ESD for its base and A for its vertex.

∴ pyramid S-$AED \Leftrightarrow$ pyramid S-ABC, § 572

(*having equal bases SED and ABC and the same altitude*).

∴ the three pyramids into which the prism ABC-SED is divided are equivalent.

∴ pyramid S-ABC is equivalent to $\frac{1}{3}$ of the prism.

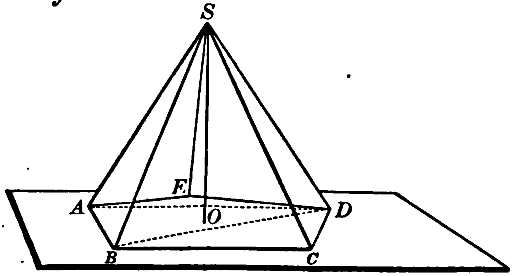
But the volume of the prism is equal to the product of its base and altitude; § 543

$\therefore S\text{-}ABC = \frac{1}{3} ABC \times H$.

Q. E. D.

PROPOSITION XVIII. THEOREM.

574. *The volume of any pyramid is equal to one-third the product of its base and altitude.*



Let S-$ABCDE$ be any pyramid.

We are to prove $S\text{-}ABCDE = \frac{1}{3} ABCDE \times SO$.

Through the edge SD, and the diagonals of the base DA, DB, pass planes.

These divide the pyramid into triangular pyramids, whose bases are the triangles which compose the base of the pyramid,

and whose common altitude is the altitude SO of the pyramid.

The volume of the given pyramid is equal to the sum of the volumes of the triangular pyramids.

But the sum of the volumes of the triangular pyramids is equal to $\frac{1}{3}$ the sum of their bases multiplied by their common altitude, § 573
(*the volume of a triangular pyramid is equal to one-third the product of its base and altitude*),

that is, the volume of the pyramid $S\text{-}ABCDE = \frac{1}{3} ABCDE \times SO$.

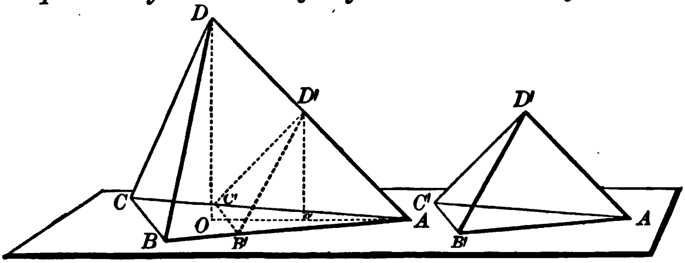
Q. E. D.

575. COROLLARY. Pyramids having equivalent bases are to each other as their altitudes; pyramids having equal altitudes are to each other as their bases. Any two pyramids are to each other as the products of their bases and altitudes.

576. SCHOLIUM. The volume of any polyhedron may be found by dividing it into pyramids, and computing the volumes of these pyramids separately.

PROPOSITION XIX. THEOREM.

577. *Two tetrahedrons having a trihedral angle of the one equal to a trihedral angle of the other are to each other as the products of the three edges of these trihedral angles.*



Let V and V' denote the volumes of the two tetrahedrons D-ABC, D'-$AB'C'$, having the trihedral A of the one equal to the trihedral A of the other.

We are to prove $\dfrac{V}{V'} = \dfrac{AB \times AC \times AD}{AB' \times AC' \times AD'}$.

Place the tetrahedrons so that their equal trihedral ∡s shall be in coincidence.

Consider ABC and $AB'C'$ the bases of the two tetrahedrons,

and from D and D' draw DO and $D'O'$ ⊥ to the base ABC.

Now $\dfrac{V}{V'} = \dfrac{ABC \times DO}{AB'C' \times D'O'} = \dfrac{ABC}{AB'C'} \times \dfrac{DO}{D'O'}$, § 575

(*any two pyramids are to each other as the products of their bases and altitudes*).

But $\dfrac{ABC}{AB'C'} = \dfrac{AB \times AC}{AB' \times AC'}$, § 341

and $\dfrac{DO}{D'O'} = \dfrac{AD}{AD'}$, § 278

(*being homologous sides of the similar* ▵s ADO *and* $AD'O'$).

$\therefore \dfrac{V}{V'} = \dfrac{AB \times AC \times AD}{AB' \times AC' \times AD'}$.

Q. E. D.

EXERCISES.

1. Given a cubical tank holding one ton of water; find its length in feet, if a cubic foot of water weigh 1000 ounces.

2. At 17 cents a square foot, what is the cost of lining with zinc a rectangular cistern 5 ft. 7 in. long, 3 ft. 11 in. broad, 2 ft. 8½ in. deep?

3. Find the side of a cubical block of cast iron weighing a ton, if iron weigh 7.2 as much as water, and a cubic foot of water weigh 1000 ounces.

4. How many cubic yards of gravel will be required for a walk surrounding a rectangular lawn 200 yards long, and 100 yards wide; the walk to be 3 feet wide and the gravel 3 inches deep?

5. The volume of a rectangular solid is the sum of two cubes whose edges are 10 inches and 2 inches respectively, and the area of its base is the difference between 2 squares whose sides are 1⅓ feet and 1⅔ feet respectively; find its altitude in feet.

6. A rectangular cistern whose length is equal to its breadth is 22 decimetres deep, and contains 10 tonneaux of water; find its length.

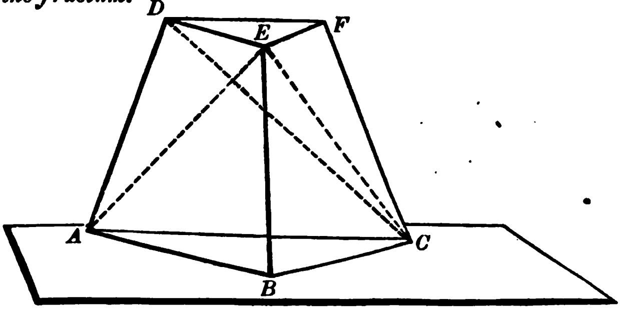
7. Given a regular prism whose base is a regular hexagon inscribed in a circle 6 metres in diameter, and whose altitude is 8.7 metres; find the number of kilolitres it will contain, if the thickness of the walls be 1 decimetre.

8. A pond whose area is 11 hectares, 21 ares, 22.2 centares, is covered with ice 21 centimetres thick. What is the weight of this body of ice in kilogrammes, the weight of ice being 92 % that of water.

9. Given two hollow oblique prisms, whose interior dimensions are as follows: the area of a right section of the first is 18 sq. ft., of the second 2.1 sq. metres; a lateral edge of the first is 9 ft., of the second 2.1 metres; find the volume of each in cubic yards, cubic metres, cubic decimetres, and cubic centimetres; find the capacity of each in gallons and litres, in bushels and hectolitres; and find the weight of water in pounds and in kilogrammes, required to fill each prism.

Proposition XX. Theorem.

578. ***The frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the two bases of the frustum.***



Let B and b denote the lower and upper bases of the frustum $ABC\text{-}DEF$, and H its altitude.

Through the vertices A, E, C and E, D, C pass planes dividing the frustum into three pyramids.

Now the pyramid $E\text{-}ABC$ has for its altitude H, the altitude of the frustum, and for its base B, the lower base of the frustum.

And the pyramid $C\text{-}EDF$ has for its altitude H, the altitude of the frustum, and for its base b, the upper base of the frustum. Hence, it only remains

To prove $E\text{-}ADC$ *equivalent to a pyramid, having for its altitude* H, *and for its base* $\sqrt{B \times b}$.

$E\text{-}ABC$ and $E\text{-}ADC$, regarded as having the common vertex C, and their bases in the same plane BD, have a common altitude.

$$\therefore E\text{-}ABC : E\text{-}ADC :: \triangle AEB : \triangle AED. \qquad \S\ 575$$

(*pyramids having equal altitudes are to each other as their bases*).

Now since the $\triangle$s AEB and AED have a common altitude,

(*that is, the altitude of the trapezoid* $ABED$),

we have $\triangle AEB : \triangle AED :: AB : DE$, § 326

$$\therefore E\text{-}ABC : E\text{-}ADC :: AB : DE.$$

In like manner $E\text{-}ADC$ and $E\text{-}DFC$, regarded as having the common vertex E and their bases in the same plane DC, have a common altitude.

$$\therefore E\text{-}ADC : E\text{-}DFC :: \triangle ADC : \triangle DFC. \qquad \S\ 575$$

But since the $\triangle$s ADC and DFC have a common altitude,
(the altitude of the trapezoid $ACFD$),

we have $\triangle ADC : \triangle DFC :: AC : DF.$ § 326

Now $\triangle DEF$ is similar to $\triangle ABC$, § 565
(the section of a pyramid made by a plane ∥ to the base is a polygon similar to the base);

$$\therefore AB : DE :: AC : DF. \qquad \S\ 278$$

$$\therefore E\text{-}ABC : E\text{-}ADC :: E\text{-}ADC : E\text{-}DFC.$$

Now $E\text{-}ABC = \frac{1}{3} H \times B,$ § 573

and $E\text{-}DFC = C\text{-}EDF = \frac{1}{3} H \times b.$ § 573

$$\therefore E\text{-}ADC = \sqrt{\tfrac{1}{3} H \times B \times \tfrac{1}{3} H \times b} = \tfrac{1}{3} H \sqrt{B \times b}.$$

Q. E. D.

579. Corollary 1. Since the volume of the frustum is denoted by V, the lower base by B, the upper base by b, and the altitude by H,

we have $V = \frac{1}{3} H \times B + \frac{1}{3} H \times b + \frac{1}{3} H \times \sqrt{B \times b}$

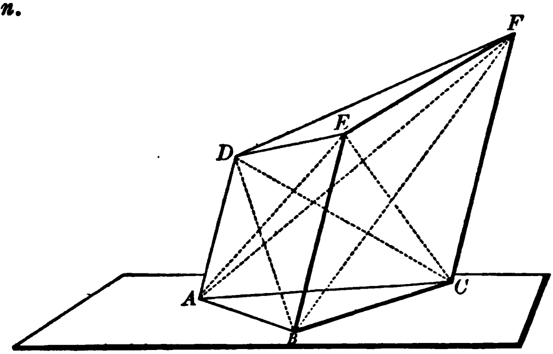
$$= \tfrac{1}{3} H \times (B + b + \sqrt{B \times b}).$$

580. Cor. 2. *The frustum of any pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*

For the frustum of any pyramid is equivalent to the corresponding frustum of a triangular pyramid having the same altitude and an equivalent base (§ 578); and the bases of the frustum of a triangular pyramid being both equivalent to the corresponding bases of the given frustum, a mean proportional between the triangular bases is equivalent to a mean proportional between their equivalents.

PROPOSITION XXI. THEOREM.

581. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism, and whose vertices are the three vertices of the inclined section.*



Let $ABC\text{-}DEF$ be a truncated triangular prism whose base is ABC, and inclined section DEF.

We are to prove $ABC\text{-}DEF \Leftrightarrow$ *three pyramids,* $E\text{-}ABC$, $D\text{-}ABC$ *and* $F\text{-}ABC$.

Pass the planes AEC and DEC, dividing the truncated prism into the three pyramids $E\text{-}ABC$, $E\text{-}ACD$, and $E\text{-}CDF$.

Now the pyramid $E\text{-}ABC$ has the base ABC and the vertex E.

$$E\text{-}ACD \Leftrightarrow B\text{-}ACD, \qquad \text{§ 574}$$

(for they have the same base ACD and the same altitude, since their vertices E and B are in the line $EB \parallel$ to the base ACD).

But pyramid $B\text{-}ACD$, which is equivalent to pyramid $E\text{-}ACD$, may be regarded as having the base ABC and the vertex D.

Again, $$E\text{-}CDF \Leftrightarrow B\text{-}ACF,$$

for their bases CDF and ACF, in the same plane, are equivalent, § 325

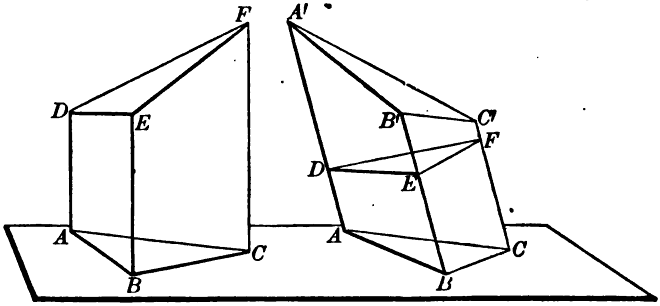
(for the △ CDF and ACF have the common base CF and equal altitudes, their vertices lying in the line $AD \parallel$ to CF).

Moreover, $E\text{-}CDF$ and $B\text{-}ACF$ have the same altitude, (*since their vertices* E *and* B *are in the line* $EB \parallel$ *to the plane of their bases* $ACDF$).

But the pyramid $B\text{-}ACF$ may be regarded as having the base ABC and the vertex F.

$\therefore$ the truncated triangular prism $ABC\text{-}DEF$ is equivalent to the three pyramids $E\text{-}ABC$, $D\text{-}ABC$, and $F\text{-}ABC$.

Q. E. D.



582. COROLLARY 1. The volume of a truncated *right* triangular prism is equal to the product of its base by one-third the sum of its lateral edges. For the lateral edges DA, EB, FC, being perpendicular to the base, are the altitudes of the three pyramids whose sum is equivalent to the truncated prism. And, since the volume of a pyramid is one-third the product of its base by its altitude, the sum of the volumes of these pyramids $= ABC \times \frac{1}{3}(DA + EB + FC)$.

583. COR. 2. The volume of *any* truncated triangular prism is equal to the product of its right section by one-third the sum of its lateral edges.

For let $ABC\text{-}A'B'C'$ be any truncated triangular prism. Then the right section DEF divides it into two truncated right prisms whose volumes are $DEF \times \frac{1}{3}(AD + BE + CF)$ and $DEF \times \frac{1}{3}(A'D + B'E + C'F)$.

Whence their sum is $DEF \times \frac{1}{3}(AA' + BB' + CC')$.

EXERCISES.

1. Given a pyramid whose base is a rectangle 80 feet by 60 feet, and whose lateral edges are each 130 feet; find its volume, and its entire surface.

2. Given the frustum of a pyramid whose bases are heptagons; each side of the lower base being 10 feet, and of the upper base 6 feet, and the slant height 42 feet; find the convex surface in square yards.

3. Given a stick of timber 30 feet long, the greater end being 18 inches square, and the smaller end 15 inches square; find its volume in cubic feet.

4. Given a stone obelisk in the form of a regular quadrangular pyramid, having a side of its base equal to 25 decimetres, and its slant height 12 metres. The stone weighs 2.5 as much as water. What is its weight in kilogrammes?

5. Given the frustum of a pyramid whose bases are squares; each side of the lower base being 35 decimetres, each side of the upper base 25 decimetres, and the altitude 15 metres; find its volume in steres.

6. Given a right hexagonal pyramid whose base is inscribed in a circle 30 feet in diameter, and whose altitude is 20 feet; find its convex surface, and its volume.

7. Given a right pentagonal pyramid whose base is inscribed in a circle 20 feet in diameter, and whose slant height is 30 feet; find its convex surface, and its volume.

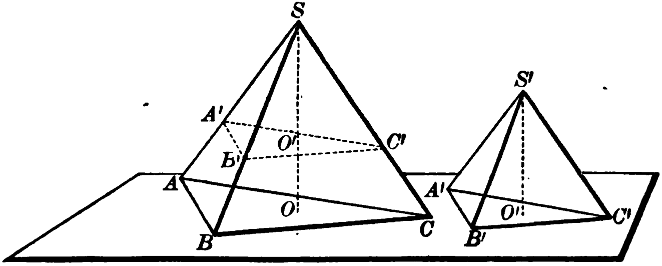
8. Find the difference between the volume of the frustum of a pyramid, and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them.

9. Given a stick of timber 32 feet long, 18 inches wide, 15 inches thick at one end, and 12 inches at the other; find the number of cubic feet, and the number of feet *board measure* it contains. Find equivalents for the results in the metric system.

ON SIMILAR POLYHEDRONS.

584. DEF. *Similar polyhedrons* are polyhedrons which have the same form. They have, therefore, the same number of faces, respectively similar and similarly placed, and their corresponding polyhedral angles equal.

585. DEF. *Homologous faces, lines,* and *angles* of similar polyhedrons are faces, lines, and angles similarly placed.



I. *The homologous edges of similar polyhedrons are proportional.*

Since the faces SAB, SAC, SBC and ABC are similar respectively to $S'A'B'$, $S'A'C'$, $S'B'C'$ and $A'B'C'$, we have

$$\frac{SA}{S'A'} = \frac{SB}{S'B'} = \frac{AB}{A'B'}, \text{ etc.} \qquad \S\ 278$$

II. *Any two homologous faces of similar polyhedrons are proportional to the squares of any two homologous edges.*

$$\text{Thus, } \frac{SAB}{S'A'B'} = \frac{\overline{SA}^2}{\overline{S'A'}^2} = \frac{SAC}{S'A'C'} = \frac{\overline{SC}^2}{\overline{S'C'}^2} = \frac{SBC}{S'B'C'}. \qquad \S\ 342$$

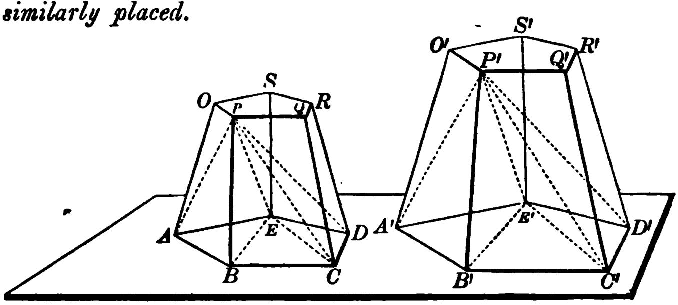
III. *The entire surfaces of two similar polyhedrons are proportional to the squares of any two homologous edges.*

$$\text{Thus, since } \frac{SAB}{S'A'B'} = \frac{SAC}{S'A'C'}, \text{ etc.,}$$

$$\frac{SAB + SAC, \text{ etc.}}{S'A'B' + S'A'C', \text{ etc.}} = \frac{SAB}{S'A'B'} = \frac{\overline{SA}^2}{\overline{S'A'}^2}. \qquad \S\ 266$$

PROPOSITION XXII. THEOREM.

586. *Two similar polyhedrons may be decomposed into the same number of tetrahedrons similar, each to each, and similarly placed.*



Let $ABCDE$-$OPQRS$ **and** $A'B'C'D'E'$-$O'P'Q'R'S'$ **be two similar polyhedrons of which** P **and** P' **are homologous vertices.**

We are to prove that $ABCDE$-$OPQRS$ *and* $A'B'C'D'E'$-$O'P'Q'R'S'$ *can be decomposed into the same number of tetrahedrons similar and similarly placed.*

Place these polyhedrons in the same plane, having any two homologous edges, as AB and $A'B'$ ‖ and lying in the same direction.

On any two corresponding faces not adjacent to P and P', as $ABCDE$ and $A'B'C'D'E'$, from two homologous vertices, as E and E', draw diagonals dividing these faces into △, similar and similarly placed.

From the homologous vertices P, P' of the polyhedrons draw straight lines to the vertices of these △.

Repeat this construction for each of the faces not adjacent to P, P'.

Then the polyhedrons will be divided into the same number of tetrahedrons;

that is, into as many tetrahedrons as there are △ in these faces.

Now, any two corresponding tetrahedrons, as $P\text{-}ABE$ and $P'\text{-}A'B'E'$, are similar;

for the faces EAB and PAB are similar respectively to the faces $E'A'B'$ and $P'A'B'$, § 294

(being similarly situated $\triangle$ *of similar polygons).*

In the $\triangle$ PBE and $P'B'E'$

PB is $\parallel$ to $P'B'$, and BE to $B'E'$,

(since they make equal $\angle$ *respectively with the* $\parallel$ *lines* AB *and* $A'B'$*);*

$$\therefore \angle PBE = \angle P'B'E', \qquad \S\ 462$$

(two $\angle$ *not in the same plane having their sides* $\parallel$ *and lying in the same direction are equal);*

and
$$\frac{PB}{P'B'} = \left(\frac{AB}{A'B'}\right) = \frac{BE}{B'E'}. \qquad \S\ 278$$

$\therefore$ face PBE is similar to face $P'B'E'$. § 284

Also, in the $\triangle$ PAE and $P'A'E'$

$$\frac{PE}{P'E'} = \left(\frac{PB}{P'B'}\right) = \frac{PA}{P'A'} = \left(\frac{AB}{A'B'}\right) = \frac{AE}{A'E'}, \qquad \S\ 278$$

(being homologous sides of similar $\triangle$*).*

$\therefore$ face PAE is similar to face $P'A'E'$. § 282

Moreover, since any two corresponding trihedral $\angle$ of these tetrahedrons are formed by three plane $\angle$ which are equal, each to each, and similarly situated, they are equal. § 492

$\therefore$ $P\text{-}ABE$ and $P'\text{-}A'B'E'$ are similar. § 584

In like manner we may show that any other two tetrahedrons similarly situated are similar.

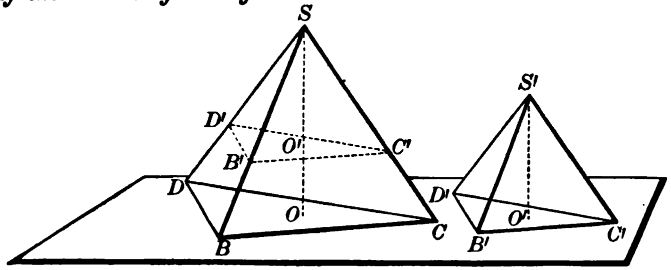
That is, the two similar polyhedrons have the same number of tetrahedrons similar each to each, and similarly situated.

Q. E. D.

587. Corollary. Any two homologous lines in two similar polyhedrons have the same ratio as any two homologous edges.

PROPOSITION XXIII. THEOREM.

588. *Similar tetrahedrons are to each other as the cubes of their homologous edges.*



Let $S\text{-}BCD$ and $S'\text{-}B'C'D'$ be two similar tetrahedrons having for bases the similar faces BCD and $B'C'D'$, and for altitudes SO and $S'O'$.

We are to prove $$\frac{S\text{-}BCD}{S'\text{-}B'C'D'} = \frac{\overline{BC}^3}{\overline{B'C'}^3}.$$

Apply the tetrahedron $S'\text{-}B'C'D'$ to the tetrahedron $S\text{-}BCD$, so that the polyhedral S' shall coincide with S.

Then the base $B'C'D'$ will be ∥ to the face BCD,

(*since their planes make equal $\angle$s with the face SBC*),

and the ⊥ SO, ⊥ to BCD, will also be ⊥ to $B'C'D'$.

SO' will be the altitude of the tetrahedron $S\text{-}B'C'D'$.

Now $$\frac{S\text{-}BCD}{S\text{-}B'C'D'} = \frac{BCD \times SO}{B'C'D' \times SO'} = \frac{BCD}{B'C'D'} \times \frac{SO}{SO'},$$ § 575

(*any two tetrahedrons are to each other as the products of their bases and altitudes*).

Since the bases are similar,

$$\frac{BCD}{B'C'D'} = \frac{\overline{BC}^2}{\overline{B'C'}^2}.$$ § 343

Also, $$\frac{SO}{SO'} = \frac{BC}{B'C'},$$ § 587

(in two similar polyhedrons any two homologous lines are in the same ratio as any two homologous edges).

$$\therefore \frac{S\text{-}BCD}{S\text{-}B'C'D'} = \frac{\overline{BC}^2}{\overline{B'C'}^2} \times \frac{BC}{B'C'} = \frac{\overline{BC}^3}{\overline{B'C'}^3}.$$

Q. E. D.

589. Corollary 1. *Two similar polyhedrons are to each other as the cubes of any two homologous edges.*

For, two similar polyhedrons may be decomposed into tetrahedrons similar, each to each, and similarly placed, of which any two homologous edges have the same ratio as any two homologous edges of the polyhedrons. And, since any pair of the similar tetrahedrons are to each other as the cubes of any two homologous edges, the entire polyhedrons are to each other as the cubes of any two homologous edges. § 266

590. Cor. 2. *Similar prisms or pyramids are to each other as the cubes of their altitudes; and similar polyhedrons are to each other as the cubes of any two homologous lines.*

Ex. 1. The portion of a tetrahedron cut off by a plane parallel to any face is a tetrahedron similar to the given tetrahedron.

Ex. 2. Two tetrahedrons, having a dihedral angle of one equal to a dihedral angle of the other, and the faces including these angles respectively similar, and similarly placed, are similar.

Ex. 3. Given two similar polyhedrons, whose volumes are 125 feet and 12.5 feet respectively; find the ratio of two homologous edges.

ON REGULAR POLYHEDRONS.

591. DEF. A *Regular polyhedron* is a polyhedron all of whose faces are equal regular polygons, and all of whose polyhedral angles are equal.

The regular polyhedrons are the *tetrahedron*, *octahedron* and *icosahedron*, all of whose faces are equal equilateral triangles; the *hexahedron*, or *cube*, whose faces are squares; the *dodecahedron*, whose faces are regular pentagons.

Only these five regular polyhedrons are possible, for a polyhedral angle must have at least three face angles, and must have the sum of its face angles less than four right angles, (§ 488). Hence:

I. If the faces be *equilateral triangles*, polyhedral angles may be formed of them in groups of 3, 4, or 5 only, as in the *tetrahedron*, *octahedron* and *icosahedron*. Since each angle of an equilateral triangle is two-thirds of a right angle, the sum of six such angles is four right angles, and therefore greater than a convex polyhedral angle.

II. If the faces be *squares*, polyhedral angles may be formed of them in groups of three only, as in the regular *hexahedron*, or *cube*; since four such angles would be four right angles.

III. If the faces be regular *pentagons*, polyhedral angles may be formed of them in groups of three only, as in the regular *dodecahedron*; since four such angles would be greater than four right angles.

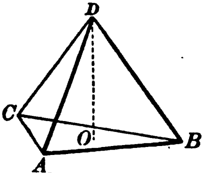
IV. We can proceed no farther; for a group of three angles of regular hexagons would equal four right angles, and of regular heptagons, etc., would be greater than four right angles.

PROPOSITION XXIV. PROBLEM.

592. *Given an edge, to construct the five regular polyhedrons.*

Let A B be the given edge.

I. *Upon A B to construct a regular tetrahedron.*



Upon $A B$ construct the equilateral $\triangle$ $A B C$. § 232

Find the centre O of this $\triangle$, § 238

and erect $O D \perp$ to the plane $A B C$.

Take the point D so that $A D = A B$.

Draw $D A, D B, D C$.

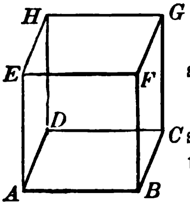
$A B C D$ is the regular tetrahedron required.

For, the edges are all equal, § 450

and hence the faces are equal equilateral $\triangle$s.

and its polyhedral $\angle$s are all equal. § 492

II. *To construct a regular hexahedron.*

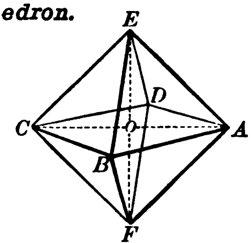


Upon the given edge $A B$ construct the square $A B C D$,

and upon the sides of this square construct the squares $E B$, $F C$, $G D$, $H A \perp$ to the plane $A B C D$.

Then $A G$ is the regular hexahedron required.

III. *To construct a regular octahedron.*



Upon the given edge $A B$ construct the square $A B C D$.

Through its centre O pass a $\perp$ to its plane $A B C D$.

In this $\perp$ take two points E and F, one above and the other below the plane, so that $A E$ and $A F$ are each equal to $A B$.

Join E and F to each of the vertices of the square.

Then $E A B C D F$ is the regular octahedron required.

For, the edges are all equal, § 450

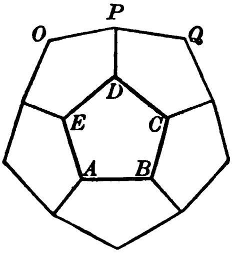
and hence the faces are equal equilateral $\triangle$s.

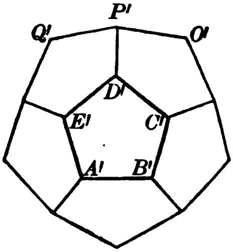
And, since the $\triangle$s $D E F$ and $D A C$ are equal, § 108

$D E B F$ is a square and the pyramid A-$D E B F$ is equal in all its parts to the pyramid E-$A B C D$.

Hence, the polyhedral $\angle$s A and E are equal.

In like manner all the polyhedral $\angle$s of the figure are equal.





IV. ***To construct a regular dodecahedron.***

Upon A B construct the regular pentagon $A\,B\,C\,D\,E$. § 395

On each side of this pentagon construct an equal pentagon, so inclined that trihedral ∠s shall be formed at A, B, C, D, E.

The convex surface thus formed is composed of six regular pentagons.

In like manner, upon an equal pentagon $A'\,B'\,C'\,D'\,E'$ construct an equal convex surface.

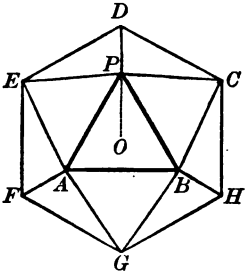
Apply one of these surfaces to the other, with their convexities turned in opposite directions, so that $P'\,O'$ and $P'\,Q'$ shall fall upon $P\,O$ and $P\,Q$.

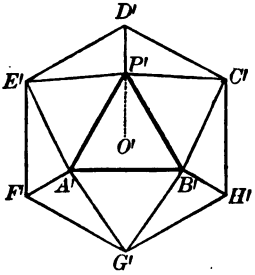
Then every face ∠ of the one will, with two consecutive face ∠s of the other, form a trihedral ∠.

The solid thus formed is the regular dodecahedron required.

For, the faces are all regular pentagons, Cons.

and the polyhedral ∠s are all equal. § 492





V. ***To construct a regular icosahedron.***

Upon $A\,B$ construct the regular pentagon $A\,B\,C\,D\,E$. § 395

At its centre O erect a ⊥ to its plane.

In this ⊥ take P so that $P\,A = A\,B$.

Join P with each of the vertices of the pentagon;

thus forming a regular pentagonal pyramid whose vertex is P, and whose dihedral $\angle$s formed on the edges PA, PB, PC, etc. are all equal. § 571

Taking A and B as vertices, construct two pyramids each equal to the first, and having for bases $BPEFG$ and $AGHCP$ respectively.

There will thus be formed a convex surface consisting of ten equal equilateral $\triangle$s.

In like manner upon an equal pentagon $A'B'C'D'E'$ construct an equal convex surface.

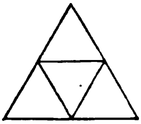
Apply one of these surfaces to the other with their convexities turned in opposite directions, so that every combination of two face $\angle$s of the one, as $P'D'C'$, $P'D'E'$, shall with a combination of three face $\angle$s of the other, as BCH, BCP, PCD, form a pentahedral $\angle$.

The solid thus formed is the regular icosahedron required.

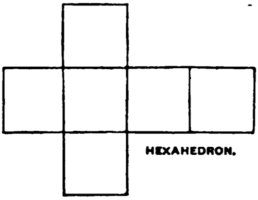
For, the faces are all equal; Cons.

and the polyhedral $\angle$s are all equal, § 571

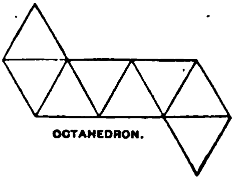
Q. E. D.



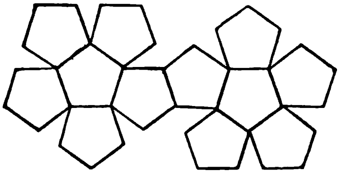
TETRAHEDRON.



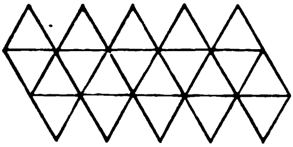
HEXAHEDRON.



OCTAHEDRON.



DODECAHEDRON.



ICOSAHEDRON.

593. Scholium. The regular polyhedrons can be formed thus:

Draw the above diagrams upon card-board. Cut through the exterior lines and half through the interior lines. The figures will then readily bend into the regular forms required.

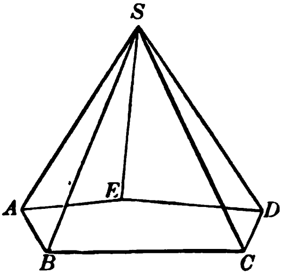
SUPPLEMENTARY PROPOSITIONS.

PROPOSITION XXV. THEOREM. (EULER'S.)

594. *In any polyhedron the number of its edges increased by two is equal to the number of its vertices increased by the number of its faces.*

Let E denote the number of edges of any polyhedron; V the number of its vertices, F the number of its faces.

We are to prove $E + 2 = V + F$.



Beginning with one face $ABCDE$, we have $E = V$.

Annex a second face SAB by applying one of its edges to an edge of the first face.

There is formed a surface having one edge AB, and two vertices A and B common to both faces.

∴ whatever the number of the sides of the new face, the whole number of edges is now one more than the whole number of vertices.

$$\therefore \text{for 2 faces } E = V + 1.$$

Annex a third face, SBC, adjacent to each of the former.

The new surface will have two edges SB and BC,

and three vertices S, B and C, in common with the preceding surface.

∴ the increase in the number of edges is again one more than the increase in the number of vertices.

According to the same law, for an incomplete surface of $F - 1$ faces

$$E = V + F - 2.$$

When we add the last face SEA, necessary to complete the surface,

its edges SE, SA and AE, and its vertices S, E and A will be in common with the preceding surface.

$$\therefore \text{in a polyhedron of } F \text{ faces } E = V + F - 2.$$

$$\therefore E + 2 = V + F.$$

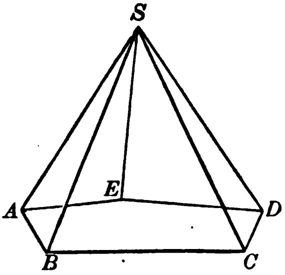
Q. E. D.

PROPOSITION XXVI. THEOREM.

595. *The sum of all the angles of the faces of any polyhedron is equal to four right angles taken as many times as the polyhedron has vertices less two.*

Let E denote the number of edges, V the number of vertices, F the number of faces, and S the sum of all the angles of the faces of any polyhedron.

We are to prove $S = 4 \text{ rt. } \measuredangle s \times (V - 2)$.



Since E denotes the number of the edges of the polyhedron,

$2E$ will denote the whole number of sides of all its faces, considered as sides of independent polygons.

And since the sum of all the interior and exterior $\measuredangle s$ of each polygon is equal to 2 rt. $\measuredangle s$ taken as many times as it has sides,

the sum of the interior and exterior $\measuredangle s$ of all the faces is equal to $2 \text{ rt. } \measuredangle s \times 2E$.

And since the sum of the exterior $\measuredangle s$ of each face is 4 rt. $\measuredangle s$, § 159

the sum of the exterior $\measuredangle s$ of all the faces is equal to $4 \text{ rt. } \measuredangle s \times F$.

$$\therefore S + 4 \text{ rt. } \measuredangle s \times F = 2 \text{ rt. } \measuredangle s \times 2E.$$

That is, $S = 4 \text{ rt. } \measuredangle s \times (E - F)$.

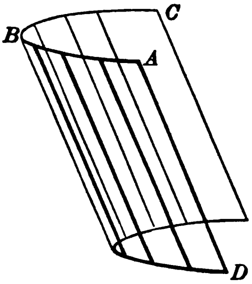
Since $E + 2 = V + F$, § 594

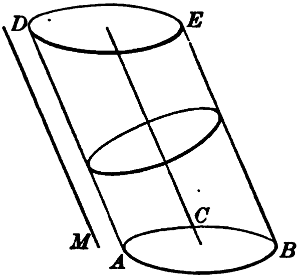
$$E - F = V - 2,$$

$$\therefore S = 4 \text{ rt. } \measuredangle s \times (V - 2).$$ Q. E. D.

ON THE CYLINDER.

596. Def. A *Cylindrical* surface is a curved surface generated by a moving straight line which continually touches a given curve and in all its positions is parallel to a given fixed straight line not in the plane of the curve.





Thus, the surface *A B C D*, generated by the moving line *A D* continually touching the curve *A B C* and always parallel to a given straight line *M*, is a cylindrical surface.

597. Def. The moving line is called the *Generatrix;* the curve which directs the motion of the generatrix is called the *Directrix;* the generatrix in any position is called an *Element* of the surface.

The generatrix may be indefinite in extent, and the directrix a closed or an open curve. In elementary geometry the directrix is considered a circle.

598. Def. *A Cylinder* is a solid bounded by a cylindrical surface and two parallel planes.

599. Def. The *Bases* of a cylinder are its plane surfaces.

600. Def. The *Lateral surface* of a cylinder is its cylindrical surface.

601. Def. The *Axis* of a cylinder is the straight line joining the centres of its bases.

602. Def. The *Altitude* of a cylinder is the perpendicular distance between the planes of its bases.

603. Def. A *Section* of a cylinder is a plane figure whose boundary is the intersection of its plane with the surface of the cylinder.

604. Def. A *Right section* of a cylinder is a section perpendicular to the elements.

605. Def. A *Radius* of a cylinder is the radius of the base.

606. Def. A *Right* cylinder is a cylinder whose elements are perpendicular to its bases. Any element of a right cylinder is equal to its altitude.

607. Def. An *Oblique* cylinder is a cylinder whose elements are oblique to its bases. Any element of an oblique cylinder is greater than its altitude.

608. Def. A *Cylinder of Revolution* is a cylinder generated by the revolution of a rectangle about one side as an axis.

609. Def. *Similar* cylinders of revolution are cylinders generated by similar rectangles revolving about homologous sides.

610. Def. A *Tangent line* to a cylinder is a straight line which touches the surface of the cylinder, but does not intersect it.

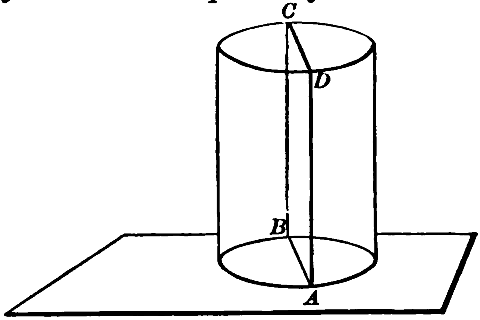
611. Def. A *Tangent plane* to a cylinder is a plane which embraces an element of the cylinder. The element embraced by the tangent plane is called the *Element of Contact.*

612. Def. A prism is *inscribed* in a cylinder when its lateral edges are elements of the cylinder and its bases are inscribed in the bases of the cylinder.

613. Def. A prism is *circumscribed* about a cylinder when its lateral faces are tangent to the cylinder and its bases are circumscribed about the bases of the cylinder.

PROPOSITION XXVII. THEOREM.

614. ***Every section of a cylinder made by a plane passing through an element is a parallelogram.***



Let $A\,B\,C\,D$ be a section of the cylinder $A\,C$, made by a plane passing through $A\,D$.

We are to prove the section $A\,B\,C\,D$ a parallelogram.

The line $B\,C$, in which the cutting plane intersects the curved surface a second time, is an element;

for, if through the point B a line be drawn ∥ to $A\,D$,

it will be an element of the surface.

It will also lie in the plane $A\,C$,
(*for its extremities lie in the plane*).

This element, lying in both the cylindrical surface and plane surface, is their intersection.

Now $A\,D$ is ∥ to $B\,C$,
(*being elements of the cylinder*),

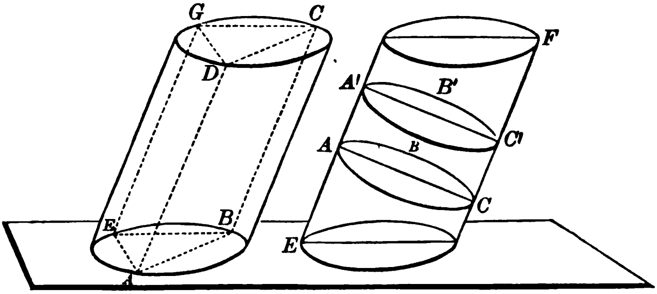
and $A\,B$ is ∥ to $D\,C$, § 465
(*the intersections of two* ∥ *planes by a third plane are* ∥ *lines*).

∴ the section $A\,B\,C\,D$ is a ▱. § 125
Q. E. D.

615. COROLLARY. Every section of a right cylinder embracing an element is a rectangle.

PROPOSITION XXVIII. THEOREM.

616. *The bases of a cylinder are equal.*



Let ABE and DCG be the bases of the cylinder AC.

We are to prove $ABE = DCG$.

Any sections AC and AG, embracing AD, an element of the cylinder, are ▱s. § 614

$\therefore AB = DC$ and $AE = DG$. § 134

Now BC is $\parallel$ to EG, § 459
(*each being* $\parallel$ *to* AD).

Also $BC = EG$, § 464

$\therefore EC$ is a ▱. § 136

$\therefore EB = GC$, § 134

$\therefore \triangle EAB = \triangle GDC$. § 108

Apply the upper base to the lower base, so that DC will coincide with AB.

Then $\triangle GDC$ will coincide with $\triangle EAB$, and point G will fall upon point E.

That is, any point G in the perimeter of the upper base will coincide with the point in the same element in the lower base.

$\therefore$ the bases coincide, and are equal.

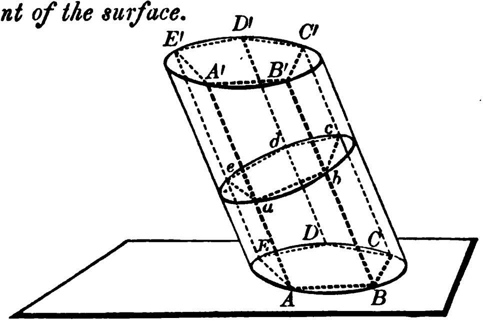
Q. E. D.

617. COROLLARY 1. Any two parallel sections ABC and $A'B'C'$, cutting all the elements of a cylinder EF, are equal. For these sections are the bases of the cylinder AC'.

618. COR. 2. Any section of a cylinder parallel to the base is equal to the base.

PROPOSITION XXIX. THEOREM.

619. *The lateral area of a cylinder is equal to the product of the perimeter of a right section of the cylinder by an element of the surface.*



Let $A B C D E$ be the base, and $A A'$ any element of the cylinder $A C'$; and let the curve $a b c d e$ be any right section of its surface.

Denote the perimeter of the right section by P,
and the lateral surface of the cylinder by S.

We are to prove $S = P \times A A'$.

Inscribe in the cylinder a prism whose right section $a b c d e$ will be a regular polygon inscribed in the right section $a b c d e$ of the cylinder. § 604

Denote the lateral area of the prism by s,
and the perimeter of its right section by p.

Then $s = p \times A A'$, § 524

(the lateral area of a prism is equal to the product of the perimeter of a right section by a lateral edge).

Now let the number of lateral faces of the inscribed prism be indefinitely increased,

the new edges continually bisecting the arcs in the right section.

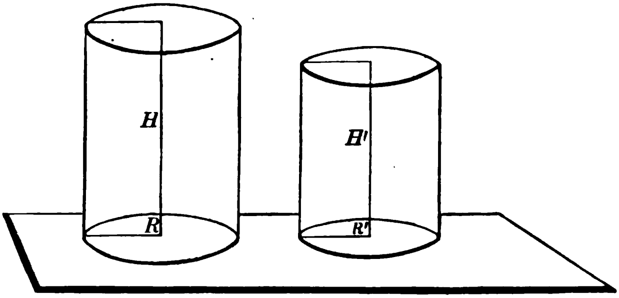
Then p approaches P as its limit,
and s approaches S as its limit.

But, however great the number of faces,

$$s = p \times A A'.$$

$$\therefore S = P \times A A', \qquad \S\ 199$$

Q. E. D.



620. Corollary 1. *The lateral area of a right cylinder is equal to the product of the perimeter of its base by its altitude.*

621. Cor. 2. Let a cylinder of revolution be generated by the rectangle whose sides are R and H revolving about the side H.

Then R is the radius of the base of the cylinder, and H the altitude of the cylinder.

The perimeter of the base is $2\pi R$; § 381

hence, $S = 2\pi R \times H.$

The area of each base is πR^2; § 381

hence, the total area T of a cylinder of revolution is expressed by

$$T = 2\pi R \times H + 2\pi R^2 = 2\pi R\,(H + R).$$

622. Cor. 3. Let S, S' denote the lateral areas of two similar cylinders of revolution;

T, T' their total areas; R, R' the radii of their bases; H, H' their altitudes.

Since the generating rectangles are similar, we have

$$\frac{H}{H'} = \frac{R}{R'} = \frac{H+R}{H'+R'}. \qquad \text{§ 266}$$

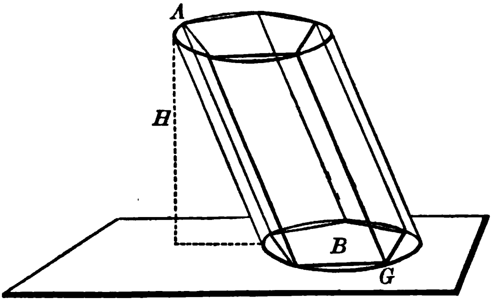
$$\therefore \frac{S}{S'} = \frac{2\pi R H}{2\pi R' H'} = \frac{R}{R'} \times \frac{H}{H'} = \frac{H^2}{H'^2} = \frac{R^2}{R'^2};$$

and $$\frac{T}{T'} = \frac{2\pi R\,(H+R)}{2\pi R'\,(H'+R')} = \frac{R}{R'}\left(\frac{H+R}{H'+R'}\right) = \frac{H^2}{H'^2} = \frac{R^2}{R'^2}.$$

That is, *the lateral areas, or the total areas, of similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases.*

PROPOSITION XXX. THEOREM.

623. *The volume of a cylinder is equal to the product of its base by its altitude.*



Let V denote the volume of the cylinder AG, B its base, and H its altitude.

We are to prove $V = B \times H$.

Let V' denote the volume of the inscribed prism AG, B' its base, and H will be its altitude.

Then $V' = B' \times H$, § 543

(*the volume of a prism is equal to the product of its base by its altitude*).

Now, let the number of lateral faces of the inscribed prism be indefinitely increased, the new edges continually bisecting the arcs of the bases.

Then B' approaches B as its limit,

and V' approaches V as its limit.

But however great the number of the lateral faces,

$$V' = B' \times H.$$

$$\therefore V = B \times H. \qquad \text{§ 199}$$

Q. E. D.

624. COROLLARY 1. Let V be the volume of a cylinder of revolution, R the radius of its base, and H its altitude.

Then the area of its base is πR^2, § 381

$$\therefore V = \pi R^2 \times H.$$

625. COR. 2. Let V and V' be the volumes of two similar cylinders of revolution, R and R' the radii of their bases, H and H' their altitudes.

Since the generating rectangles are similar, we have

$$\frac{H}{H'} = \frac{R}{R'};$$

and $$\frac{V}{V'} = \frac{\pi R^2 H}{\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{H^3}{H'^3} = \frac{R^3}{R'^3}.$$

That is, *the volumes of similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

Ex. 1. Required, the entire surface and volume of a cylinder of revolution whose altitude is 30 inches, and whose base is a circle of which the diameter is 20 inches.

2. Required, the volume of a right truncated triangular prism the area of whose base is 40 inches, and whose lateral edges are 10, 12, and 15 inches, respectively.

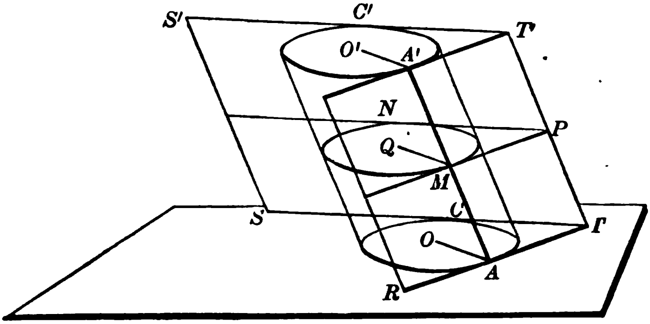
3. Let E denote an edge of a regular tetrahedron; show that the altitude of the tetrahedron is equal to $E\sqrt{\frac{2}{3}}$; that the surface is equal to $E^2\sqrt{3}$; and that the volume is equal to $\frac{E^3}{12}\sqrt{2}$.

4. Required, the number of quarts that a cylinder of revolution will contain whose height is 20 inches, and whose diameter is 12 inches.

5. Given S, the surface of a cube, find its edge, diagonal, and volume. What do these become when $S = 54$?

PROPOSITION XXXI. PROBLEM.

626. *Through a given point to pass a plane tangent to a given cylinder.*



CASE I. — *When the given point is in the curved surface of the cylinder.*

Let $A C'$ be a given cylinder, and let the given point be a point in the element $A A'$.

It is required to pass a plane tangent to the cylinder and embracing the element $A A'$.

Draw the radius $O A$, and $A T$ tangent to the base;

and pass a plane $R T'$ through $A A'$ and $A T$.

The plane $R T'$ is the plane required.

For, through any point P in this plane, not in the element $A A'$,

pass a plane ∥ to the base, intersecting the cylinder in the ⊙ $M N$,

and the plane $R T'$ in $M P$.

From the centre of the ⊙ $M N$ draw $Q M$.

$M P$ and $M Q$ are ∥ respectively to $A T$ and $A O$, § 465

(*the intersections of two* ∥ *planes by a third plane are* ∥ *lines*);

$\therefore \angle P M Q = \angle T A O$, § 462

(*two* ∠s *not in the same plane, having their sides* ∥ *and lying in the same direction, are equal*).

$\therefore PM$ is tangent to the $\odot MN$ at M. § 186

$\therefore P$ lies without the $\odot MN$,

and hence without the cylinder.

$\therefore$ the plane RT' does not cut the cylinder, and is tangent to it.

Case II. — *When the given point is without the cylinder.*

Let P be the given point.

It is required to pass a plane through P tangent to the cylinder.

Through P draw the line $PT \parallel$ to the elements of the cylinder,

meeting the plane of the base at T.

From T draw TA and TC tangents to the base. § 240

Through PT and the tangent TA pass a plane RT'.

Since AA' is $\parallel$ to PT, Cons.

the plane RT', passing through PT and the point A will contain the element AA',

(*two* $\parallel$ *lines lie in the same plane*).

And, since RT' also contains the tangent AT,

it is a tangent plane to the cylinder.

In like manner, the plane TS', passed through PT and the tangent line TC,

is a tangent plane to the cylinder.

Q. E. F.

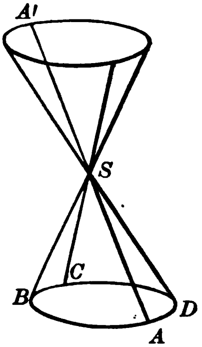
627. Corollary 1. The intersection of two tangent planes to a cylinder is parallel to the elements of the cylinder.

628. Cor. 2. Any straight line drawn in a tangent plane, and cutting the element of contact, is tangent to the cylinder.

ON THE CONE.

629. DEF. A *Conical* surface is a surface generated by a moving straight line continually touching a given curve and passing through a fixed point not in the plane of the curve.

Thus the surface generated by the moving line $A A'$ continually touching the curve $ABCD$, and passing through the fixed point S, is a conical surface.



630. DEF. The moving line is called the *Generatrix;* the curve which directs the motion of the generatrix is called the *Directrix;* the generatrix, in any position, is called an *Element* of the surface.

631. DEF. A conical surface generated by an indefinite straight line consists of two portions, called *Nappes,* one the *Lower,* the other the *Upper Nappe.*

632. DEF. A *Cone* is a solid bounded by a conical surface and a plane.

633. DEF. The *Lateral* surface of a cone is its conical surface.

634. DEF. The *Base* of a cone is its plane surface.

635. DEF. The *Vertex* of a cone is the fixed point through which all the elements pass.

636. DEF. The *Altitude* of a cone is the perpendicular distance between its vertex and the plane of its base.

637. DEF. The *Axis* of a cone is the straight line joining its vertex and the centre of its base.

638. DEF. A *Section* of a cone is a plane figure whose boundary is the intersection of its plane with the surface of the cone.

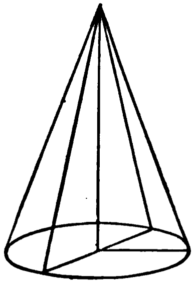
639. DEF. A *Right section* of a cone is a section perpendicular to the axis.

640. DEF. A *Circular* cone is a cone whose base is a circle.

641. DEF. A *Right* cone is a cone whose axis is perpendicular to its base. The axis of a right cone is equal to its altitude.

642. DEF. An *Oblique* cone is a cone whose axis is oblique to its base. The axis of an oblique cone is greater than its altitude.

643. Def. A *Cone of Revolution* is a cone generated by the revolution of a right triangle about one of its perpendicular sides as an axis.



The side about which the triangle revolves is the axis of the cone; the other perpendicular generates the *base*, the hypotenuse generates the *conical surface*. Any position of the hypotenuse is an *element*, and any element is called the *slant height*.

644. Def. *Similar* cones of revolution are cones generated by the revolution of similar right triangles about homologous perpendicular sides.

645. Def. A *Truncated* cone is the portion of a cone included between the base and a section cutting all the elements.

646. Def. A *Frustum* of a cone is a truncated cone in which the cutting section is parallel to the base.

647. Def. The base of the cone is called the *Lower* base of the frustum, and the parallel section the *Upper* base.

648. Def. The *Altitude* of a frustum is the perpendicular distance between the planes of its bases.

649. Def. The *Lateral surface* of a frustum is the portion of the lateral surface of the cone included between the bases of the frustum.

650. Def. The *Slant* height of a frustum of a cone of revolution is the portion of any element of the cone included between the bases.

651. Def. A *Tangent line* to a cone is a line having only one point in common with the surface.

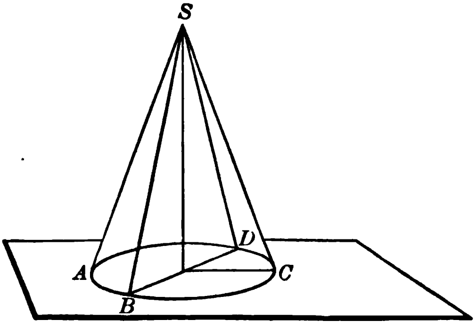
652. Def. A *Tangent plane* to a cone is a plane embracing an element of the cone without cutting the surface. The element embraced by the tangent plane is called the *Element of Contact.*

653. Def. A pyramid is *inscribed* in a cone when its lateral edges are elements of the cone and its base is inscribed in the base of the cone.

654. Def. A pyramid is *circumscribed* about a cone when its lateral faces are tangent to the cone and its base is circumscribed about the base of the cone.

Proposition XXXII. Theorem.

655. *Every section of a cone made by a plane passing through its vertex is a triangle.*



Let SBD be a section of the cone S-ABC through the vertex S.

We are to prove the section SBD a triangle.

The straight lines joining S with B and D are elements of the surface. § 630

They also lie in the cutting plane,
(*for their extremities lie in the plane*).

Hence, they are the intersections of the conical surface with the plane of the section.

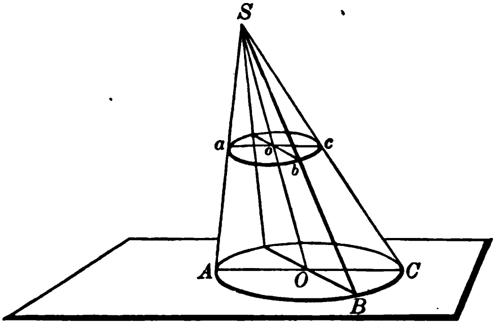
BD is also a straight line, § 446
(*the intersection of two planes is a straight line*).

∴ the section SBD is a △.

Q. E. D.

PROPOSITION XXXIII. THEOREM.

656. *Every section of a circular cone made by a plane parallel to the base is a circle.*



Let the section $a\,b\,c$ of the circular cone $S\text{-}A\,B\,C$ be parallel to the base.

We are to prove that $a\,b\,c$ is a circle.

Let O be the centre of the base, and let o be the point in which the axis $S\,O$ pierces the plane of the ∥ section.

Through $S\,O$ and any number of elements, $S\,A$, $S\,B$, etc., pass planes cutting the base in the radii $O\,A$, $O\,B$, etc.,

and the section $a\,b\,c$ in the straight lines $o\,a$, $o\,b$, etc.

Now $o\,a$ and $o\,b$ are ∥ respectively to $O\,A$ and $O\,B$, § 465
(*the intersections of two ∥ planes by a third plane are ∥ lines*).

∴ the ⚠ $S\,o\,a$ and $S\,o\,b$ are similar respectively to the ⚠ $S\,O\,A$ and $S\,O\,B$, § 279

and their homologous sides give the proportion

$$\frac{o\,a}{O\,A} = \left(\frac{S\,o}{S\,O}\right) = \frac{o\,b}{O\,B}.$$

But $O\,A = O\,B$; § 163

$$\therefore o\,a = o\,b.$$

That is, all the straight lines drawn from o to the perimeter of the section are equal.

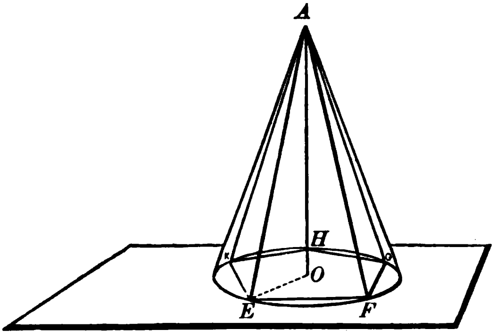
∴ the section $a\,b\,c$ is a ⊙.

Q. E. D.

657. COROLLARY. The axis of a circular cone passes through the centres of all the sections which are parallel to the base.

PROPOSITION XXXIV. THEOREM.

658. *The lateral area of a cone of revolution is equal to one-half the product of the circumference of its base by the slant height.*



Let A-$EFGHK$ be a cone generated by the revolution of the right triangle AOE about AO as an axis, and let S denote its lateral area, C the circumference of its base and L its slant height.

We are to prove $S = \frac{1}{2} C \times L$.

Inscribe on the base any regular polygon $EFGHK$,

and upon this polygon as a base construct the regular pyramid A-$EFGHK$ inscribed in the cone.

Denote the lateral area of this pyramid by s, the perimeter of its base by p, its slant height by l,

Then $s = \frac{1}{2} p \times l$, § 569

(the lateral area of a regular pyramid is equal to one-half the product of the perimeter of its base by the slant height).

Now, let the number of the lateral faces of the inscribed pyramid be indefinitely increased, the new edges continually bisecting the arcs of the base.

Then p, s and l approach C, S and L respectively as their limits.

But however great the number of lateral faces of the pyramid,

$$s = \tfrac{1}{2} p \times l.$$

$$\therefore S = \tfrac{1}{2} C \times L. \qquad \S\ 199$$

Q. E. D.

659. Corollary 1. If R be the radius of the base, we have $C = 2\pi R$ (§ 381). Therefore $S = \frac{1}{2}(2\pi R \times L) = \pi R L$. Also, since the area of the base is πR^2, the total area T of the cone is expressed by

$$T = \pi R L + \pi R^2 = \pi R (L + R).$$

660. Cor. 2. Let S and S' denote the lateral areas of two similar cones of revolution, T and T' their total areas, R and R' the radii of their bases, H and H' their altitudes, L and L' their slant heights. Since the generating triangles are similar, we have

$$\frac{L}{L'} = \frac{H}{H'} = \frac{R}{R'} = \frac{R + L}{R' + L'}. \qquad \S\ 266$$

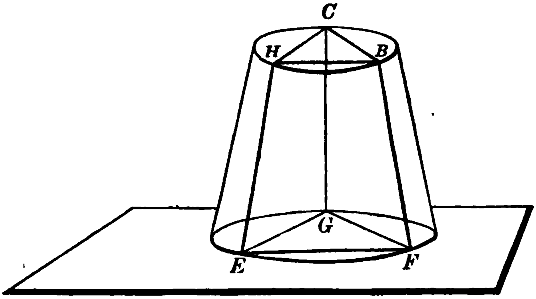
$$\therefore \frac{S}{S'} = \frac{\pi R L}{\pi R' L'} = \frac{R}{R'} \times \frac{L}{L'} = \frac{L^2}{L'^2} = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}.$$

And $\dfrac{T}{T'} = \dfrac{\pi R \times (L + R)}{\pi R' \times (L' + R')} = \dfrac{R}{R'} \times \dfrac{L + R}{L' + R'} = \dfrac{L^2}{L'^2} = \dfrac{R^2}{R'^2} = \dfrac{H^2}{H'^2}.$

That is: *the lateral areas*, or *total areas*, *of similar cones of revolution are to each other as the squares of their slant heights, the squares of their altitudes, or the squares of the radii of their bases.*

PROPOSITION XXXV. THEOREM.

661. *The lateral area of the frustum of a cone of revolution is equal to one-half the sum of the circumferences of its bases multiplied by the slant height.*



Let HBC-EFG be the frustum of a cone of revolution, and let S denote its lateral area, C and c the circumferences of its lower and upper bases, R and r the radii of the bases, and L the slant height.

We are to prove $S = \frac{1}{2}(C + c) \times L.$

Inscribe in the frustum of the cone the frustum of the regular pyramid HBC-EFG,

and denote the lateral area of this frustum by s, the perimeters of its lower and upper bases by P and p respectively, and its slant height by l.

Then $s = \frac{1}{2}(P + p)\,l,$ § 570

(*the lateral area of the frustum of a regular pyramid is equal to one-half the sum of the perimeters of its bases multiplied by the slant height*).

Now, let the number of lateral faces be indefinitely increased, the new elements constantly bisecting the arcs of the bases.

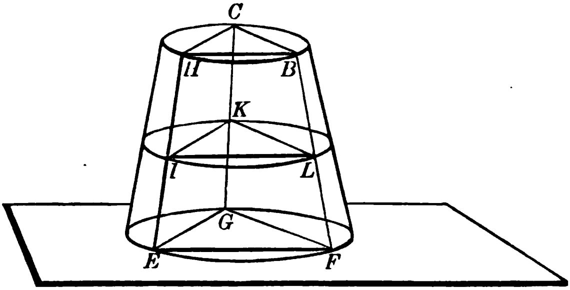
Then P, p, and l, approach C, c, and L, respectively as their limits.

But, however great the number of lateral faces of the frustum of the pyramid,

$$s = \tfrac{1}{2}(P + p) \times l.$$

$$\therefore S = \tfrac{1}{2}(C + c) \times L. \qquad \text{§ 199}$$

Q. E. D.



662. Corollary. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equidistant from its bases multiplied by its slant height.*

For the section of the frustum equidistant from its bases cuts the frustum of the regular inscribed pyramid equidistant from its bases.

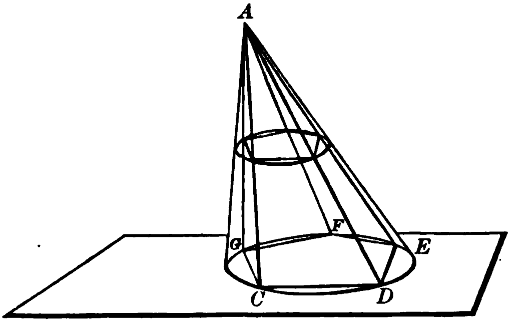
Therefore the perimeter $ILK = \frac{1}{2}$ the sum of the perimeters HBC and EFG. § 142

And this will always be true, however great the number of the lateral faces of the frustum of the pyramid.

Hence, circumference $ILK = \frac{1}{2}$ the sum of the circumferences HBC and EFG. § 199

PROPOSITION XXXVI. THEOREM.

663. *Any section of a cone parallel to the base is to the base as the square of the altitude of the part above the section is to the square of the altitude of the cone.*



Let B denote the base of the cone, H its altitude, b a section of the cone parallel to the base, and h the altitude of the cone above the section.

We are to prove $B : b :: H^2 : h^2$.

Let B' denote the base of an inscribed pyramid, b' the base of the pyramid formed in the section of the cone.

Then $B' : b' :: H^2 : h^2$, § 566

(any section of a pyramid ∥ to its base is to the base as the square of the ⊥ from the vertex to the plane of the section is to the square of the altitude of the pyramid).

Now let the number of lateral faces of the inscribed pyramid be indefinitely increased,

the new edges continually bisecting the arcs in the base of the cone.

Then B' and b' approach B and b respectively as their limits.

But however great the number of lateral faces of the pyramid,

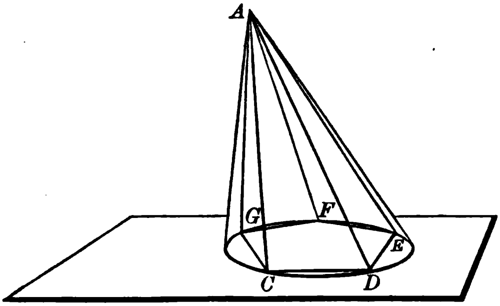
$$B' : b' :: H^2 : h^2.$$

$$\therefore B : b :: H^2 : h^2,$$ § 199

Q. E. D.

PROPOSITION XXXVII. THEOREM.

664. *The volume of any cone is equal to the product of one-third of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the cone.

We are to prove $V = \frac{1}{3} B \times H$.

Let the volume of an inscribed pyramid A-$CDEFG$ be denoted by V', and its base by B'.

H will also be the altitude of this pyramid.

Then $V' = \frac{1}{3} B' \times H$, § 574

Now, let the number of lateral faces of the inscribed pyramid be indefinitely increased, the new edges continually bisecting the arcs in the base of the cone.

Then V' approaches to V as its limit, and B' to B as its limit.

But however great the number of lateral faces of the pyramid,

$$V' = \frac{1}{3} B' \times H.$$

$$\therefore V = \frac{1}{3} B \times H. \qquad \text{§ 199}$$

Q. E. D.

665. COROLLARY 1. If the cone be a cone of revolution, and R be the radius of the base, then $B = \pi R^2$ (§ 381); $\therefore V = \frac{1}{3} \pi R^2 \times H$.

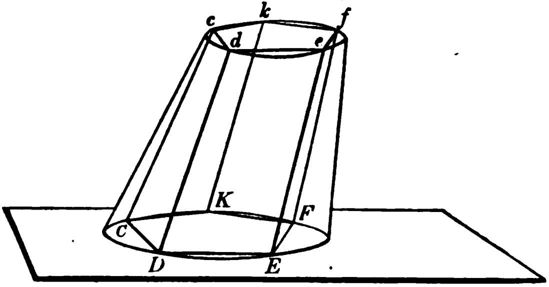
666. COR. 2. *Similar cones of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.* For, let R and R' be the radii of two similar cones of revolution, H and H' their altitudes, V and V' their volumes. Since the generating triangles are similar, we have

$$H : H' :: R : R'.$$

$$\therefore \frac{V}{V'} = \frac{\frac{1}{3} \pi R^2 \times H}{\frac{1}{3} \pi R'^2 \times H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{H^3}{H'^3} = \frac{R^3}{R'^3}.$$

PROPOSITION XXXVIII. THEOREM.

667. *A frustum of any cone is equivalent to the sum of three cones whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*



Let V denote the volume of the frustum, B its lower base, b its upper base, and H its altitude.

We are to prove $V = \frac{1}{3} H (B + b + \sqrt{B \times b})$.

Let V' denote the volume of an inscribed frustum of a pyramid, B' its lower base, b' its upper base.

Its altitude will also be H.

Then, $V' = \frac{1}{3} H (B' + b' + \sqrt{B' \times b'})$, § 578

(*a frustum of any pyramid is* ⇔ *to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum*).

Now, let the number of lateral faces of the inscribed frustum be indefinitely increased,

the new edges continually bisecting the arcs in the bases of the frustum of the cone.

But however great the number of lateral faces of the frustum of the pyramid,

$$V' = \frac{1}{3} H (B' + b' + \sqrt{B' \times b'}.$$

$$\therefore V = \frac{1}{3} H (B + b + \sqrt{B \times b}). \qquad \text{§ 199}$$

Q. E. D.

668. COROLLARY. If the frustum be that of a cone of revolution, and R and r be the radii of its bases, we have $B = \pi R^2$, and $b = \pi r^2$,

and $\sqrt{B \times b} = \pi R r$.

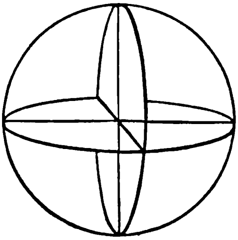
$$\therefore V = \frac{1}{3} \pi H (R^2 + r^2 + R r).$$

BOOK VIII.

THE SPHERE.

ON SECTIONS AND TANGENTS.

669. DEF. A *Sphere* is a solid bounded by a surface all points of which are equally distant from a point within called the *centre.* A sphere may be generated by the revolution of a semicircle about its diameter as an axis.



670. DEF. A *Radius* of a sphere is the distance from its centre to any point in the surface. All the radii of a sphere are equal.

671. DEF. A *Diameter* of a sphere is any straight line passing through the centre and having its extremities in the surface of the sphere. All the diameters of a sphere are equal, since each is equal to twice the radius.

672. DEF. A *Section* of a sphere is a plane figure whose boundary is the intersection of its plane with the surface of the sphere.

673. DEF. A line or plane is *Tangent to a sphere* when it has one, and only one, point in common with the surface of the sphere.

674. DEF. Two spheres are *tangent to each other* when their surfaces have one, and only one, point in common.

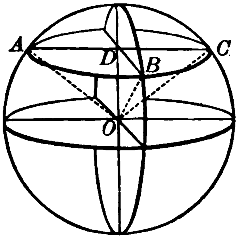
675. DEF. A *polyhedron is circumscribed* about a sphere when all of its faces are tangent to the sphere. In this case the sphere is inscribed in the polyhedron.

676. DEF. A *polyhedron is inscribed* in a sphere when all of its vertices are in the surface of the sphere. In this case the sphere is *circumscribed about* the polyhedron.

677. DEF. A *Cylinder* or *cone is circumscribed* about a sphere when its bases and cylindrical surface, or its base and conical surface, are tangent to the sphere. In this case the sphere is *inscribed* in the cylinder or cone.

PROPOSITION I. THEOREM.

678. *Every section of a sphere made by a plane is a circle.*



Let the section A B C be a plane section of a sphere whose centre is O.

We are to prove section A B C a circle.

From the centre *O* draw *O D* ⊥ to the section, and draw the radii *O A*, *O B*, *O C*, to different points in the boundary of the section.

In the rt. ▲ *O D A*, *O D B* and *O D C*,

O D is common, and *O A*, *O B* and *O C* are equal,
(being radii of the sphere).

∴ the rt. ▲ *O DA*, *O D B* and *O D C* are equal, § 109
(two rt. ▲ are equal when they have a side and hypotenuse of the one equal respectively to a side and hypotenuse of the other).

∴ *D A*, *D B* and *D C* are equal,
(being homologous sides of equal ▲).

∴ the section *A B C* is a circle whose centre is *D*.
Q. E. D.

679. COROLLARY 1. The line joining the centres of a sphere and a circle of a sphere is perpendicular to the circle.

680. COR. II. If R, r and p, respectively, denote the radius of a sphere, the radius of a circle of a sphere, and the perpendicular from the centre of the sphere to the circle, then $r = \sqrt{R^2 - p^2}$. Therefore all circles of a sphere equally distant from the centre are equal, and of two circles unequally distant from the centre of the sphere the more remote is the smaller.

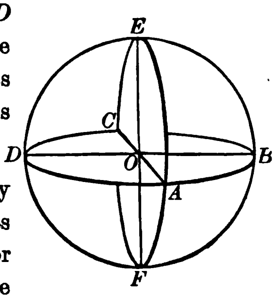
Again, if $p = 0$, then $r = R$, and the centre of the sphere and the centre of the circle coincide; such a section is the greatest possible circle of the sphere.

681. Def. A *Great circle* of a sphere is a section of the sphere made by a plane passing through the centre.

682. Def. A *Small circle* of a sphere is a section of the sphere made by a plane not passing through the centre.

683. Def. An *Axis* of a circle of a sphere is the diameter of the sphere *perpendicular* to the circle; and the extremities of the axis are the *Poles* of the circle.

684. *Every great circle bisects the sphere.* For, if the parts be separated and placed with their plane sections in coincidence and their convexities turned the same way, their convex surfaces will coincide; otherwise there would be points in the spherical surface unequally distant from the centre.



685. Any two great circles, $ABCD$ and $AECF$, bisect each other. For the intersection AC of their planes passes through the centre of the sphere, and is a diameter of each circle.

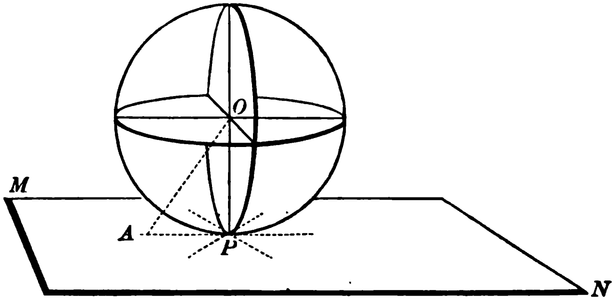
686. An arc of a great circle may be drawn through any two given points A and E in the surface of a sphere. For the two points A and E, and the centre O, determine the plane of a great circle whose circumference passes through A and E. § 443

If, however, the two given points are the extremities A and C of the diameter of the sphere, the position of the circle is not determined. For, the points A, O and C, being in the same straight line, an infinite number of planes can pass through them. § 441

687. An arc of a circle may be drawn through any three given points on the surface of a sphere. For, the three points determine the plane which cuts the sphere in a circle.

PROPOSITION II. THEOREM.

688. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*



Let O be the centre of a sphere, and MN a plane perpendicular to the radius OP, at its extremity P.

We are to prove MN *tangent to the sphere.*

From O draw any other straight line OA to the plane MN.

$$OP < OA, \qquad \S\ 448$$

(a ⊥ is the shortest distance from a point to a plane).

∴ point A is without the sphere.

But OA is any other line than OP,

∴ every point in the plane MN is without the sphere, except P.

∴ MN is tangent to the sphere at P. § 673

Q. E. D.

689. COROLLARY 1. A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.

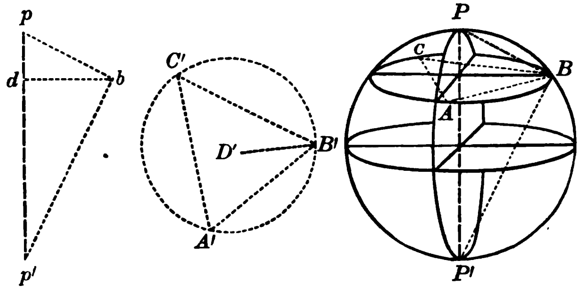
690. COR. 2. A straight line tangent to a circle of a sphere lies in a plane tangent to the sphere at the point of contact.

691. COR. 3. Any straight line in a tangent plane through the point of contact is tangent to the sphere at that point.

692. COR. 4. The plane of any two straight lines tangent to the sphere at the same point is tangent to the sphere at that point.

PROPOSITION III. PROBLEM.

693. *Given a material sphere to find its diameter.*



Let $PBP'C$ represent a material sphere.

It is required to find its diameter.

From any point P of the given surface, with any opening of the compasses, describe the circumference ABC on the surface.

Then the rectilinear line PB, being the opening of the compasses, is a known line.

Take any three points A, B and C in this circumference, and with the compasses measure the rectilinear distances AB, BC and CA.

Construct the △ $A'B'C'$, with its sides equal respectively to AB, BC and CA. § 232

Circumscribe a circle about the △ $A'B'C'$. § 239

The radius $D'B'$ of this ⊙ is equal to the radius of ⊙ ABC.

Construct the rt. △ bdp, having the hypotenuse $bp = BP$, and one side $bd = B'D'$.

Draw $bp' \perp$ to bp, and meeting pd produced in p'.

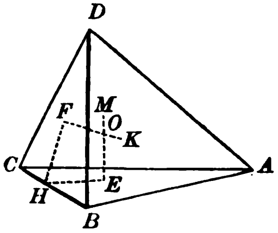
Then pp' is equal to the diameter of the given sphere.

For, if we bisect the sphere through P and B, and in the section draw the diameter PP' and chord BP', the △ bpp', when applied to △ BPP', will coincide with it.

Q. E. F.

PROPOSITION IV. THEOREM.

694. *Through any four points not in the same plane, one spherical surface can be made to pass, and but one.*



Let A, B, C, D, be four points not in the same plane.

We are to prove that one, and only one, spherical surface can be made to pass through A, B, C, D.

Construct the tetrahedron $ABCD$, having for its vertices A, B, C, D.

Let E be the centre of the circle circumscribed about the face ABC.

Draw $EM \perp$ to this face.

Every point in EM is equally distant from the points A, B and C, § 450

(oblique lines drawn from a point to a plane at equal distances from the foot of the $\perp$ are equal).

Also, let F be the centre of the circle circumscribed about the face BCD;

and draw $FK \perp$ to this face.

Let H be the middle point of BC.

Draw EH and FH.

Then EH and FH are $\perp$ to BC. § 184

$\therefore$ the plane passed through EH and FH is $\perp$ to BC, § 449
(if a straight line be $\perp$ to two straight lines drawn through its foot in a plane, it is $\perp$ to the plane, and in this case the plane is $\perp$ to the line).

Hence, this plane is also $\perp$ to each of the faces ABC and BCD, § 471
(if a straight line be $\perp$ to a plane, every plane passed through that line is $\perp$ to the plane).

$\therefore$ the $\perp$s EM and FK lie in the plane EHF.

Hence they must meet unless they be parallel.

But if they were $\parallel$, the planes BCD and ABC would be one and the same plane, which is contrary to the hypothesis.

Now O, the point of intersection of the $\perp$s EM and FK, is equally distant from A, B and C; and also equally distant from B, C and D;

$\therefore$ it is equally distant from A, B, C and D.

Hence, a spherical surface, whose centre is O, and radius OA, will pass through the four given points.

Only one spherical surface can be made to pass through the points A, B, C and D.

For the centre of such a spherical surface must lie in both the $\perp$s EM and FK.

And, since O is the only point common to these $\perp$s, O is the centre of the only spherical surface passing through A, B, C and D.

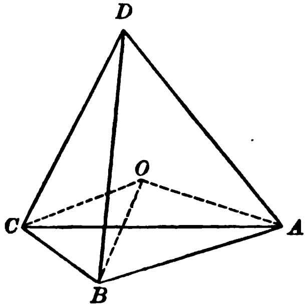
Q. E. D.

695. Corollary 1. The four perpendiculars erected at the centres of the faces of a tetrahedron meet at the same point.

696. Cor. 2. The six planes perpendicular to the six edges of a tetrahedron at their middle point will intersect at the same point.

PROPOSITION V. THEOREM.

697. *A sphere may be inscribed in any given tetrahedron.*



Let $ABCD$ be the given tetrahedron.

We are to prove that a sphere may be inscribed in $ABCD$.

Bisect the dihedral ⦣ at the edges AB, BC and AC by the planes OAB, OBC and OAC respectively.

Every point in the plane OAB is equally distant from the faces ABC and ABD, § 477

For a like reason, every point in the plane OBC is equally distant from the faces ABC and DBC;

and every point in the plane OAC is equally distant from the faces ABC and ADC.

∴ O, the common intersection of these three planes, is equally distant from the four faces of the tetrahedron.

∴ a sphere described with O as a centre, and with the radius equal to the distance of O to any face, will be tangent to each face, and will be inscribed in the tetrahedron. § 673

Q. E. D.

698. COROLLARY. The six planes which bisect the six dihedral angles of a tetrahedron intersect in the same point.

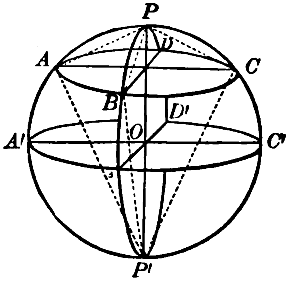
ON DISTANCES MEASURED ON THE SURFACE OF THE SPHERE.

699. DEF. The distance between two points on the surface of a sphere is understood to be the arc of a *great circle* joining the points, unless otherwise stated.

700. DEF. The distance from the pole of a circle to any point in the circumference of the circle is the *Polar distance* of the circle.

Proposition VI. Theorem.

701. *The distances measured on the surface of a sphere from all points in the circumference of a circle to its pole are equal.*



Let P, P' be the poles of the circle ABC.

We are to prove arcs PA, PB, PC equal.

The *rectilinear lines* PA, PB and PC are equal, § 450
(*oblique lines drawn from a point to a plane at equal distances from the foot of the* ⊥ *are equal*);

∴ the *arcs* PA, PB and PC are equal, § 182
(*in equal* ⓢ *equal chords subtend equal arcs*).

In like manner arcs $P'A$, $P'B$ and $P'C$ are equal.

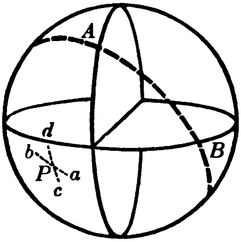
Q. E. D.

702. Corollary 1. The polar distance of a great circle is a quadrant. Thus, arcs PA', PB', $P'A'$, $P'B'$, polar distances of the great circle $A'B'C'D'$, are quadrants; for they are the measures of the right angles $A'OP$, $B'OP$, $A'OP'$, $B'OP'$, whose vertices are at the centres of the great circles $PA'P'C'$, $PB'P'D'$.

703. Scholium. Every point in the circumference of a small circle is at unequal distances from the two poles of the circle.

PROPOSITION VII. PROBLEM.

704. *To pass a circumference of a great circle through any two points on the surface of a sphere.*



Let A and B be any two points on the surface of a sphere.

It is required to pass a circumference of a great circle through A and B.

From *A* as a pole, with an arc equal to a quadrant, strike an arc *a b*,

and from *B* as a pole, with the same radius, describe an arc *c d*, intersecting *a b* at *P*.

Then a circumference described with a quadrant arc, with *P* as a pole, will pass through *A* and *B* and be the circumference of a great circle.

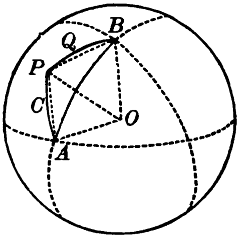
Q. E. F.

705. COROLLARY. Through any two points on the surface of a sphere, not at the extremities of the same diameter, only one circumference of a great circle can be made to pass.

706. SCHOLIUM. By means of poles arcs of circles may be drawn on the surface of a sphere with the same facility as upon a plane surface, and, in general, the methods of construction in Spherical Geometry are similar to those of Plane Geometry. Thus we may draw an arc perpendicular to a given spherical arc, bisect a given spherical angle or arc, make a spherical angle equal to a given spherical angle, etc., in the same way that we make analogous constructions in Plane Geometry.

Proposition VIII. Theorem.

707. *The shortest distance on the surface of a sphere between any two points on that surface is the arc, not greater than a semi-circumference, of the great circle which joins them.*



Let AB be the arc of a great circle which joins any two points A and B on the surface of a sphere; and let $ACPQB$ be any other line on the surface between A and B.

We are to prove arc $AB < ACPQB$.

Let P be any point in $ACPQB$.

Pass arcs of great circles through A and P, and P and B. § 704

Join A, P and B with the centre of the sphere O.

The $\angle$s AOB, AOP and POB are the face $\angle$s of the trihedral $\angle$ whose vertex is at O.

The arcs AB, AP and PB are measures of these $\angle$s. § 202

Now $\angle AOB < \angle AOP + \angle POB$, § 487

(the sum of any two face $\angle$s of a trihedral is $>$ the third $\angle$).

$\therefore$ arc $AB <$ arc $AP +$ arc PB.

In like manner, joining any point in ACP with A and P by arcs of great ⓢ, their sum would be greater than arc AP;

and, joining any point in PQB with P and B by arcs of great ⓢ, the sum of these arcs would be greater than arc PB.

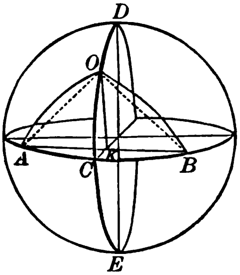
If this process be indefinitely repeated the distance from A to B on the arcs of the great ⓢ will continually increase and approach to the line $ACPQB$.

$\therefore$ arc $AB < ACPQB$.

Q. E. D.

PROPOSITION IX. THEOREM.

708. *Every point in an arc of a great circle which bisects a given arc at right angles is equally distant from the extremities of the given arc.*



Let arc CD bisect arc AB at right angles.

We are to prove any point O in CD is equally distant from A and B.

Since great circle CDE bisects arc AB at right angles, it also bisects chord AB at right angles.

Hence, chord AB is ⊥ to the plane CDE at K.

∴ OK is ⊥ to chord AB at its middle point. § 430

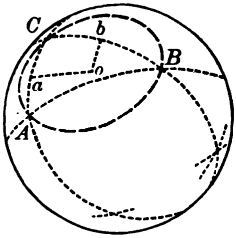
∴ *rectilinear lines* OA and OB are equal. § 58

∴ *arcs* OA and OB are equal. § 182

Q. E. D.

PROPOSITION X. PROBLEM.

709. *To pass the circumference of a small circle through any three points on the surface of a sphere.*



Let A, B and C be any three points on the surface of a sphere.

It is required to pass the circumference of a small circle through the points A, B and C.

Pass arcs of great circles through A and B, A and C, B and C. § 704

Arcs of great circles ao and bo ⊥ to AC and BC at their middle points intersect at o.

Then o is equally distant from A, B and C. § 708

∴ the circumference of a small circle drawn from o as a pole, with an arc oA will pass through A, B and C, and be the circumference required.

Q. E. D.

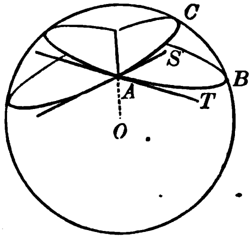
ON SPHERICAL ANGLES.

710. DEF. The *angle of two curves* which have a common point is the angle included by the two tangents to the two curves at that point.

711. DEF. A *spherical angle* is the angle included between two arcs of *great circles.*

PROPOSITION XI. THEOREM.

712. *The angle of two curves which intersect on the surface of a sphere is equal to the dihedral angle between the planes passed through the centre of the sphere, and the tangents of the two curves at their point of intersection.*



Let the curves AB and AC intersect at A on the surface of a sphere whose centre is O; and let AT and AS be the tangents to the two curves respectively.

We are to prove $\angle TAS$ *equal to the dihedral angle formed by the planes* OAT *and* OAS.

Since AT and AS do not cut the curves at A, they do not cut the surface of the sphere,

and are therefore tangents to the sphere.

$\therefore AT$ and AS are $\perp$ to the radius OA, drawn to the point of contact. § 186

$\therefore \angle TAS$ measures the dihedral $\angle$ of the planes OAT and OAS, passed through the radius OA and the tangents AT and AS. § 470

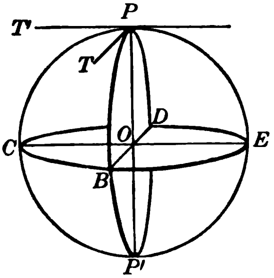
But $\angle TAS$ is the $\angle$ of the two curves AB and AC. § 710

$\therefore$ the $\angle$ of the two curves AB and $AC =$ the dihedral $\angle$ of the planes OAT and OAS.

Q. E. D.

PROPOSITION XII. THEOREM.

713. *A spherical angle is equal to the measure of the dihedral angle included by the great circles whose arcs form the sides of the angle.*



Let BPC be any spherical angle, and $BPDP'$ and $CPEP'$ the great circles whose arcs BP and CP include the angle.

We are to prove $\angle BPC$ *equal to the measure of the dihedral* $\angle C\text{-}PP'\text{-}B$.

Since two great ⓢ intersect in a diameter, PP' is a diameter. § 685

Draw PT tangent to the ⊙ $BPDP'$.

Then PT lies in the same plane as the ⊙ $BPDP'$, and is ⊥ to PP' at P.

In like manner draw PT' tangent to the ⊙ $CPEP'$.

Then PT' lies in the same plane as the ⊙ $CPEP'$, and is ⊥ to PP' at P.

∴ $\angle TPT'$ is the measure of the dihedral $\angle C\text{-}PP'\text{-}B$. § 470

But spherical $\angle BPC$ is the same as plane $\angle TPT'$; § 710

∴ spherical $\angle BPC$ is equal to the measure of dihedral $\angle C\text{-}PP'\text{-}B$.

Q. E. D.

714. COROLLARY. *A spherical angle is measured by the arc of a great circle described about its vertex as a pole and intercepted by its sides (produced if necessary).* For, if BC be the arc of a great circle described about the vertex P as a pole, PB and PC are quadrants. Hence, BO and CO are perpendicular to PP'. Therefore BOC measures the dihedral angle $B\text{-}PO\text{-}C$, and, hence, the spherical angle BPC. Therefore, arc BC, which measures the angle BOC, measures the spherical angle BPC.

ON SPHERICAL POLYGONS AND PYRAMIDS.

715. DEF. A *spherical Polygon* is a portion of a surface of a sphere bounded by three or more arcs of *great circles.*

The *sides* of a spherical polygon are the bounding arcs; the *angles* are the angles included by consecutive sides; the *vertices* are the intersections of the sides.

716. DEF. The *Diagonal* of a spherical polygon is an arc of a great circle dividing the polygon, and terminating in two vertices not adjacent.

The planes of the sides of a spherical polygon form by their intersections a polyhedral angle whose vertex is the centre of the sphere, and whose face angles are measured by the sides of the polygon.

717. DEF. A *spherical Pyramid* is a portion of a sphere bounded by a spherical polygon and the planes of the sides of the polygon.

The spherical polygon is the *base* of the pyramid, and the centre of the sphere is its *vertex.*

718. DEF. A *spherical Triangle* is a spherical polygon of three sides.

A spherical triangle, like a plane triangle, is *right,* or *oblique; scalene, isosceles* or *equilateral.*

719. DEF. Two spherical triangles are *equal* if their successive sides and angles, taken in the *same* order, be equal each to each.

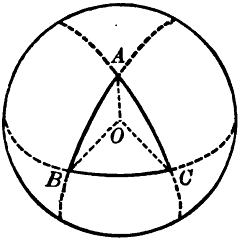
720. DEF. Two spherical triangles are *symmetrical* if their successive sides and angles, taken in *reverse* order, be equal each to each.

721. DEF. The *Polar* of a spherical triangle is a spherical triangle, the poles of whose sides are respectively the vertices of the given triangle.

Since the sides of a spherical triangle are arcs, they may be expressed in degrees and minutes.

PROPOSITION XIII. THEOREM.

722. *Any side of a spherical triangle is less than the sum of the other two sides.*



Let ABC be any spherical triangle.

We are to prove $BC < BA + AC$.

Join the vertices A, B and C with the centre O of the sphere.

Then, in the trihedral $\angle$ O-ABC thus formed, the face $\angle$s AOC, AOB and BOC are measured, respectively, by the sides AC, AB and BC. § 202

Now, $BOC < BOA + AOC$, § 487

(*the sum of any two* $\angle$s *of a trihedral is greater than the third* $\angle$).

$\therefore BC < BA + AC$.

Q. E. D.

723. COROLLARY. Any side of a spherical polygon is less than the sum of the other sides.

Ex. 1. Given a cone of revolution whose side is 24 feet, and the diameter of its base 6 feet; find its entire surface, and its volume.

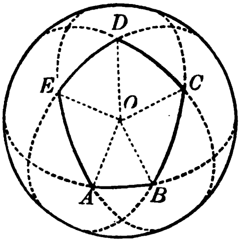
2. Given the frustum of a cone whose altitude is 24 feet, the circumference of its lower base 20 feet, and that of its upper base 16 feet; find its volume.

3. The volume of the frustum of a cone of revolution is 8025 cubic inches; its altitude 14 inches; the circumference of the lower base twice that of the upper base. What are the circumferences of the bases?

4. The frustum of a cone of revolution whose altitude is 20 feet, and the diameters of its bases 12 feet and 8 feet respectively, is divided into two equal parts by a plane parallel to its bases. What is the altitude of each part?

PROPOSITION XIV. THEOREM.

724. *The sum of the sides of a spherical polygon is less than the circumference of a great circle.*



Let $ABCDE$ be a spherical polygon.

We are to prove $AB + BC$ etc. less than the circumference of a great circle.

Join the vertices A, B, C etc., with O the centre of the sphere.

The sum of the face ⦩ AOB, BOC etc., which form a polyhedral ∠ at O, is less than four rt. ⦩. § 488

∴ the sum of the arcs AB, BC etc., which measure these face ⦩, is less than the circumference of a great circle.

Q. E. D.

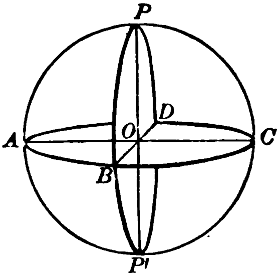
725. COROLLARY. If we denote the sides of a spherical triangle by a, b and c, then $a + b + c < 360°$.

Ex. 1. The surface of a cone is 540 square inches; what is the surface of a similar cone whose volume is 8 times as great?

2. The lateral surface of a cone is S; what is the lateral surface of a similar cone whose volume is n times as great?

PROPOSITION XV. THEOREM.

726. *A point upon the surface of a sphere, which is at the distance of a quadrant from each of two other points, is one of the poles of the great circle which passes through these points.*



Let P be a point at the distance of a quadrant from each of the two points A and B.

We are to prove P a pole of the great circle which passes through A and B.

Since PA and PB are quadrants,

$\angle$s POA and POB are rt. $\angle$s.

$\therefore PO$ is $\perp$ to the plane of the $\odot ABC$, § 449

(*a straight line $\perp$ to two straight lines drawn through its foot in a plane is $\perp$ to the plane*).

$\therefore P$ is a pole of the $\odot ABC$. § 683

Q. E. D.

Ex. 1. Show that two symmetrical polyhedrons may be decomposed into the same number of tetrahedrons symmetrical each to each.

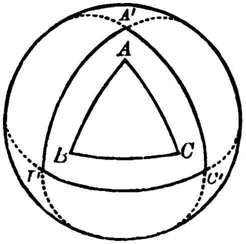
2. Show that two symmetrical polyhedrons are equivalent.

3. Show that the intersection of two planes of symmetry of a solid is an axis of symmetry.

4. Show that the intersections of three planes of symmetry of a solid are three axes of symmetry; and that the common intersection of these axes is the centre of symmetry.

PROPOSITION XVI. THEOREM.

727. *If, from the vertices of a given spherical triangle as poles, arcs of great circles be described, another triangle is formed, the vertices of which are the poles of the sides of the given triangle.*



Let $A\ B\ C$ be the given triangle; and, from its vertices A, B and C as poles, let the arcs $B'C'$, $A'C'$ and $A'B'$ respectively be described.

We are to prove vertices A', B' and C' poles respectively of arcs $B\,C$, $A\,C$ and $A\,B$.

Since B is the pole of the arc $A'C'$, and C the pole of the arc $A'B'$,

A' is at a quadrant's distance from each of the points B and C.

$\therefore$ A' is a pole of the arc $B\,C$, § 726

(*a point upon the surface of a sphere which is at a quadrant's distance from each of two other points is one of the poles of the great circle which passes through those points*).

In like manner, it may be shown that B' is a pole of the arc $A\,C$, and C' a pole of the arc $A\,B$.

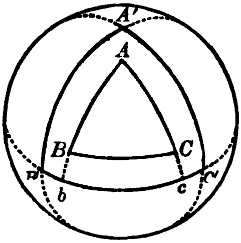
Q. E. D.

728. SCHOLIUM 1. $\triangle\ A'B'C'$ is the polar of $\triangle\ A\,B\,C$, and, reciprocally, $\triangle\ A\,B\,C$ is the polar of $\triangle\ A'B'C'$.

729. SCH. 2. The arcs of great circles described about A, B and C as poles will, if produced, form three triangles exterior to the polar. The polar triangles are distinguished by having their homologous vertices A and A' on the same side of $B\,C$ and $B'C'$, B and B' on the same side of $A\,C$ and $A'C'$, and C and C' on the same side of $A\,B$ and $A'B'$.

PROPOSITION XVII. THEOREM.

730. *In two polar triangles each angle of either is the supplement of the side lying opposite to it in the other.*



Let $A\,B\,C$ and $A'\,B'\,C'$ be two polar triangles.

We are to prove $\angle$s *A, B and C respectively the supplements of the sides B′ C′, A′ C′ and A′ B′.*

Let the sides $A\,B$ and $A\,C$, produced if necessary, meet the side $B'\,C'$ in the points b and c.

Since the vertex A is a pole of the arc $B'\,C'$, § 721

$\angle A$ is measured by $b\,c$, § 714

(*a spherical* $\angle$ *is measured by the arc of a great circle described about its vertex as a pole and intercepted by its sides*).

Now, since B' is the pole of the arc $A\,c$, $B'c = 90°$.

Since C' is the pole of the arc $A\,b$, $C'b = 90°$.

$$\therefore B'c + C'b = B'\,C' + b\,c = 180°.$$

$\therefore \angle A\ (= b\,c)$ is the supplement of the side $B'\,C'$.

In like manner it may be shown that each $\angle$ of either $\triangle$ is the supplement of the side lying opposite to it in the other.

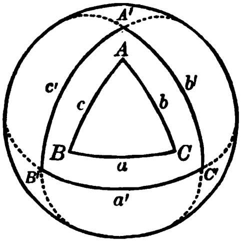
Q. E. D.

731. SCHOLIUM. *In two polar triangles each side of either is the supplement of the angle lying opposite to it in the other.* If A, B and C denote the angles, and a, b and c the sides of a triangle, the angles of the polar triangle will be $180° - a$, $180° - b$ and $180° - c$; and the sides of the polar triangle will be $180° - A$, $180° - B$ and $180° - C$.

By reason of these relations polar triangles are often called *supplemental* triangles.

PROPOSITION XVIII. THEOREM.

732. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



Let ABC be a spherical triangle.

We are to prove $\angle A + \angle B + \angle C$ *greater than* 2, *and less than* 6, *right angles.*

Denote the sides of the polar △ opposite the ∠s A, B, C respectively, by a', b', c'.

Then $\angle A = 180° - a'$, $\angle B = 180° - b'$ and $\angle C = 180° - c'$, § 730
(*in two polar* △s *each* ∠ *of either is the supplement of the side lying opposite to it in the other.*)

By adding, $\angle A + \angle B + \angle C = 540° - (a' + b' + c')$.

But $a' + b' + c'$ is less than 360°, § 724
(*the sum of the sides of a spherical polygon is less than the circumference of a great circle*).

$$\therefore \angle A + \angle B + \angle C > 180°.$$

Also, since each ∠ is less than 2 rt. ∠s,
their sum is less than 6 rt. ∠s.

Q. E. D.

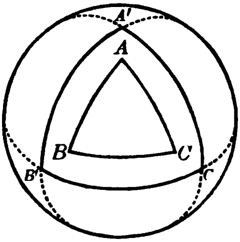
733. Corollary. A spherical triangle may have two, or even three right angles; or two, or even three obtuse angles.

734. Def. A spherical triangle having one right angle is called *rectangular;* having two right angles, *bi-rectangular;* having three right angles, *tri-rectangular.*

Each of the sides of a tri-rectangular triangle is a quadrant, and the triangle is called, when reference is had to its sides, tri-quadrantal.

PROPOSITION XIX. THEOREM.

735. *Each angle of a spherical triangle is greater than the difference between two right angles and the sum of the other two angles.*



Let $\angle$s A, B and C be the angles of the spherical triangle ABC.

We are to prove $\angle A$ *greater than the difference between* 180° *and* $(\angle B + \angle C)$.

I. Suppose $(\angle B + \angle C) < 180°$.

Now $\angle A + \angle B + \angle C > 180°$. § 732

By transposing, $\angle A > 180° - (\angle B + \angle C)$.

II. Suppose $(\angle B + \angle C) > 180°$.

Now of the three sides $(180° - \angle A)$, $(180° - \angle B)$, $(180° - \angle C)$, of the polar $\triangle$, each is less than the sum of the other two, § 722

(either side of a spherical $\triangle$ is less than the sum of the other two sides).

$\therefore (180° - \angle B) + (180° - \angle C) > 180° - \angle A;$

or, $360° - (\angle B + \angle C) > 180° - \angle A.$

By transposing, $\angle A > (\angle B + \angle C) - 180°$.

Q. E. D.

Ex. 1. The volume of a cone is 1728 cubic inches; what is the volume of a similar cone whose surface is 4 times as great?

2. The volume of a cone is V; what is the volume of a similar cone whose surface is n times as great?

736. Def. *Equal spherical triangles* are triangles which have their corresponding sides and angles equal each to each and arranged in the same order, so that when applied to each other they will coincide. Thus in Fig. 1, $A\,B\,C$ and $A'\,B'\,C'$ are equal spherical triangles.

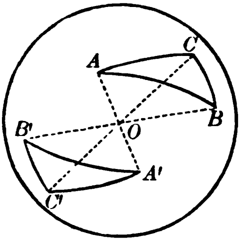


Fig. 1.

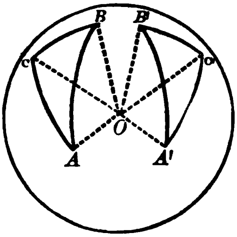


Fig. 2.

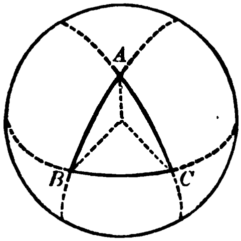
737. Def. ***Symmetrical spherical triangles*** are triangles which have their corresponding sides and angles equal each to each, but arranged in reverse order.

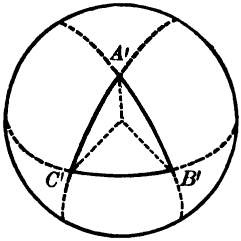
Thus, in Fig. 2, $A\,B\,C$ and $A'\,B'\,C'$ are symmetrical spherical triangles. For, since the face angles of the two trihedrals are equal respectively, but are arranged in reverse order, the *sides* of the spherical triangles, which measure these face angles, are equal, each to each, and are arranged in reverse order; and since the dihedral angles of the two trihedrals are equal respectively, but are arranged in reverse order, the *angles* of the spherical triangles, which are equal to these trihedrals, are equal, each to each, and are arranged in reverse order.

In like manner we may have symmetrical spherical polygons of any number of sides, and corresponding symmetrical spherical pyramids.

Two symmetrical spherical triangles cannot be made to coincide. For, if their convexities lie in opposite directions, they evidently will not coincide; and if their convexities lie in the same direction, and we apply $A\,B$ to $A'\,B'$, the vertices C and C' will lie on opposite sides of $A'\,B'$.

738. There is, however, one exception. *Two symmetrical isosceles spherical triangles can be made to coincide.*





Thus, if $A\,B\,C$ be an isosceles spherical triangle, $A\,B = A\,C$ and in its symmetrical triangle $A'\,B' = A'\,C'$. Hence $A\,B = A'\,C'$ and $A\,C = A'\,B'$. And, since $\angle$s A and A' are equal, if $A\,B$ be placed on $A'\,C'$, $A\,C$ will fall on its equal $A'\,B'$.

In consequence of the relations established between polyhedral angles and spherical polygons, from any property of polyhedral angles, we may infer a corresponding property of spherical polygons. Reciprocally, from any property of spherical polygons, we may infer a corresponding property of polyhedral angles.

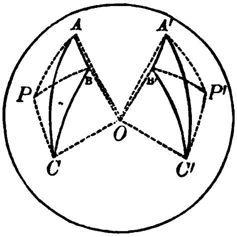
Ex. 1. The altitude of a cone of revolution is 12 inches; at what distances from the vertex must three planes be passed parallel to the base of the cone, in order to divide the lateral surface into four equal parts?

2. The altitude of a given solid is 2 inches, its surface 24 square inches, and its volume 8 cubic inches; find the altitude and surface of a similar solid whose volume is 512 cubic inches.

3. The volumes of two similar cones of revolution are 6 cubic inches and 48 cubic inches respectively, and the slant height of the first is 5 inches; find the slant height of the second.

PROPOSITION XX. THEOREM.

739. *Two symmetrical spherical triangles are equivalent.*



Let $A B C$ and $A' B' C'$ be two symmetrical spherical triangles, having $A B$, $A C$ and $B C$ equal respectively to $A' B'$, $A' C'$ and $B' C'$.

We are to prove $\triangle A B C \Leftrightarrow \triangle A' B' C'$.

Let P and P' be poles of small circles which pass through A, B, C and A', B', C'.

Now, since the arcs $A B$, $A C$ and $B C = A' B'$, $A' C'$ and $B' C'$ respectively, the *chords* of the arcs $A B$, $A C$ and $B C =$ *chords* of the arcs $A' B'$, $A' C'$ and $B' C'$ respectively. § 181

∴ the *plane* ⟁ formed by the chords of these arcs are equal. § 108

∴ Ⓢ $A B C$ and $A' B' C'$ which circumscribe these equal *plane* ⟁ are equal.

∴ the six spherical distances $P A$, $P B$, $P' A'$ etc. are equal, (*being polar distances of equal Ⓢ on the same sphere*).

∴ ⟁ $P A B$, $P' A' B'$ are symmetrical and isosceles.

So likewise are ⟁ $P B C$, $P' B' C'$ and ⟁ $P A C$, $P' A' C'$.

∴ $\triangle P A B$ may be applied to $\triangle P' A' B'$ and will coincide with it. § 738

So likewise $\triangle P B C$ with $\triangle P' B' C'$ and $\triangle P A C$ with $\triangle P' A' C'$.

∴ ⟁ $P A B + P B C - P A C \Leftrightarrow$ ⟁ $P' A' B' + P' B' C' - P' A' C'$.

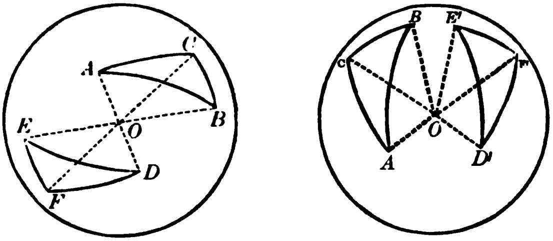
$$\therefore \triangle A B C \Leftrightarrow \triangle A' B' C'.$$

Q. E. D.

740. COROLLARY. Two symmetrical spherical pyramids are equivalent.

PROPOSITION XXI. THEOREM.

741. *On the same sphere, or equal spheres, two triangles are either equal, or symmetrical and equivalent, if two sides and the included angle of the one be respectively equal to two sides and the included angle of the other.*



In the $\triangle$s ABC and DEF, let $\angle A = \angle D$, and the sides AB and AC equal respectively the sides DE and DF.

We are to prove $\triangle$s ABC and DEF equal, or symmetrical and equivalent.

I. *When the parts of the two $\triangle$s are in the same order* as in $\triangle$s ABC and DEF,

$\triangle ABC$ can be applied to $\triangle DEF$, as in the corresponding case of plane $\triangle$s, and will coincide with it. § 106

II. *When the parts are in reverse order,* as in $\triangle$s ABC and $D'E'F'$,

construct the $\triangle DEF$ symmetrical with respect to $\triangle D'E'F'$.

Then $\triangle DEF$ will have its $\angle$s and sides equal respectively to those of the $\triangle D'E'F'$. § 737

Now in the $\triangle$s ABC and DEF,

$\angle A = \angle D$, $AB = DE$ and $AC = DF$,

and these parts are arranged in the same order.

$\therefore \triangle ABC = \triangle DEF$. Case I.

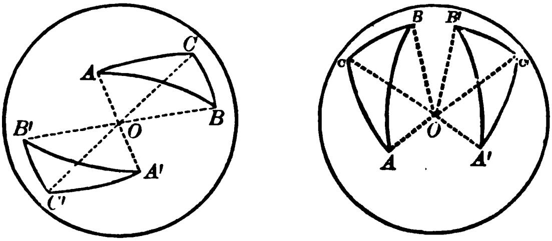
But $\triangle D'E'F' \Bumpeq \triangle DEF$, § 739

$\therefore \triangle ABC \Bumpeq \triangle D'E'F'$.

Q. E. D.

PROPOSITION XXII. THEOREM.

742. *Two triangles on the same sphere, or equal spheres, are either equal, or symmetrical and equivalent, if a side and two adjacent angles of the one be equal respectively to a side and two adjacent angles of the other.*

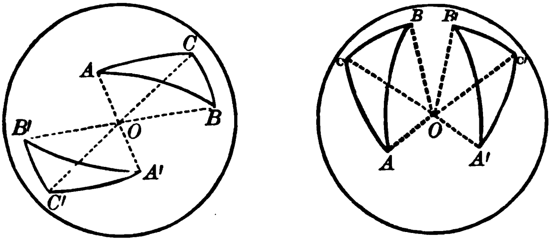


For one of the △s may be applied to the other, or to its symmetrical △, as in the corresponding case of plane △s. § 107

Q. E. D.

PROPOSITION XXIII. THEOREM.

743. *Two mutually equilateral triangles on the same sphere, or equal spheres, are mutually equiangular, and are either equal, or symmetrical and equivalent.*



For the face ∠s of the corresponding trihedral angles at the centre of the sphere are equal respectively, § 202

(*since they are measured by equal sides of the* △s).

∴ the corresponding dihedral ∠s are equal. § 492

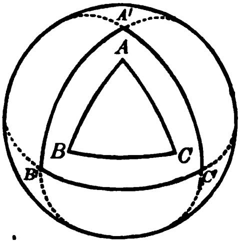
∴ the ∠s of the spherical △s are respectively equal.

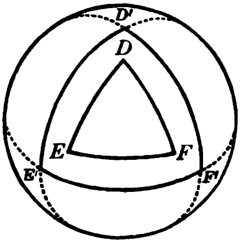
∴ the △s are either equal, or symmetrical and equivalent, according as their equal sides are arranged in the same, or reverse order.

Q. E. D.

PROPOSITION XXIV. THEOREM.

744. *Two mutually equiangular triangles on the same sphere, or equal spheres, are mutually equilateral, and are either equal, or symmetrical and equivalent.*





Let the spherical triangles $A\ B\ C$ and $D\ E\ F$ be mutually equiangular.

We are to prove △ *$A\ B\ C$ and $D\ E\ F$ mutually equilateral, and equal, or symmetrical and equivalent.*

Let △ $A'\ B'\ C'$ and $D'\ E'\ F'$ be the polar △ of △ $A\ B\ C$ and $D\ E\ F$ respectively.

Then the △ $A'\ B'\ C'$ and $D'\ E'\ F'$ are mutually equilateral, § 731

(in two polar △ each side of the one is the supplement of the ∠ lying opposite to it in the other).

∴ △ $A'\ B'\ C'$ and $D'\ E'\ F'$ are mutually equiangular, § 743

(two mutually equilateral △ on equal spheres are mutually equiangular).

∴ △ $A\ B\ C$ and $D\ E\ F$ are mutually equilateral; § 731

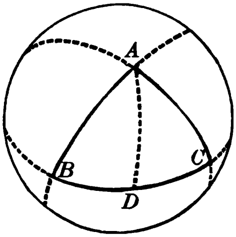
hence △ $A\ B\ C$ and $D\ E\ F$ are either equal, or symmetrical and equivalent, § 743

(two mutually equilateral △ on equal spheres are either equal, or symmetrical and equivalent).

Q. E. D.

PROPOSITION XXV. THEOREM.

745. *The angles opposite equal sides of an isosceles spherical triangle are equal.*



In the spherical $\triangle ABC$, ***let*** $AB = AC$.

We are to prove $\angle B = \angle C$.

Draw arc AD of a great circle, from the vertex A to the middle of the base BC.

Then $\triangle$ ABD and ACD are mutually equilateral.

$\therefore$ $\triangle$ ABD and ACD are mutually equiangular, § 743
(two mutually equilateral $\triangle$ *on the same sphere are mutually equiangular).*

$$\therefore \angle B = \angle C,$$

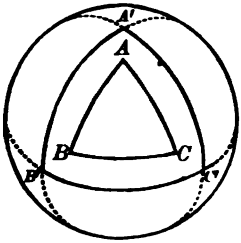
(since they are homologous $\angle$ *of symmetrical* $\triangle$*).*

Q. E. D.

746. COROLLARY. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle of the base bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetrical triangles.

Proposition XXVI. Theorem.

747. *If two angles of a spherical triangle be equal, the sides opposite these angles are equal, and the triangle is isosceles.*



***In the spherical** $\triangle ABC$, **let** $\angle B = \angle C$.*

We are to prove $AC = AB$.

Let $\triangle A'B'C'$ be the polar $\triangle$ of $\triangle ABC$.

Since $\angle B = \angle C$, Hyp.

$\therefore A'C' = A'B'$, § 731

(*in two polar* $\triangle$s *each side of one is the supplement of the* $\angle$ *lying opposite to it in the other*).

$\therefore \angle B' = \angle C'$, § 745

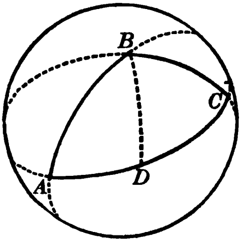
(*in an isosceles spherical* $\triangle$, *the* $\angle$s *opposite the equal sides are equal*).

$\therefore AC = AB$. § 731

Q. E. D.

PROPOSITION XXVII. THEOREM.

748. *In a spherical triangle the greater side is opposite the greater angle; and, conversely, the greater angle is opposite the greater side.*



I. ***In the*** $\triangle ABC$, ***let*** $\angle ABC > \angle C$.

We are to prove $AC > AB$.

Draw the arc BD of a great circle, making $\angle CBD = \angle C$.

Then $DC = DB$, § 747
(*if two* ∠s *of a spherical* △ *be equal the sides opposite these* ∠s *are equal*).

Add DA to each of these equals;

then $DC + DA = DB + DA$.

But $DB + DA > AB$, § 722
(*the sum of two sides of a spherical* △ *is greater than the third side*).

$\therefore DC + DA > AB$, or $AC > AB$.

II. ***Let*** $AC > AB$.

We are to prove $\angle ABC > \angle C$.

If $\angle ABC = \angle C$, $AC = AB$, § 747

and if $\angle ABC < \angle C$, $AC < AB$. Case I.

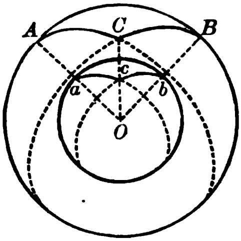
But both of these conclusions are contrary to the hypothesis.

$\therefore \angle ABC > \angle C$.

Q. E. D.

PROPOSITION XXVIII. THEOREM.

749. *On unequal spheres mutually equiangular triangles are similar.*



From O, the common centre of two unequal spheres, draw the radii OA, OB and OC cutting the surface of the smaller sphere in a, b and c. Draw arcs of great circles, AB, AC, BC, ab, ac, bc.

We are to prove $\triangle ABC$ *similar to* $\triangle abc$.

∠ A, B, C are equal respectively to ∠ a, b, c,
(since the corresponding dihedrals in each case are the same).

In the similar sectors AOB and aOb,

$AB : ab :: AO : aO$; § 385

and in the similar sectors AOC and aOc,

$AC : ac :: AO : aO$. § 385

$\therefore AB : ab :: AC : ac$.

In like manner, $AB : ab :: BC : bc$.

That is, the homologous sides of the two △ are proportional, and their homologous ∠ are equal.

$\therefore \triangle ABC$ is similar to $\triangle abc$.

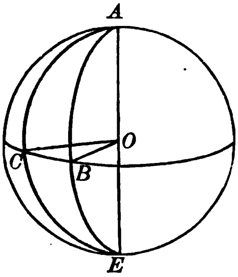
Q. E. D.

750. SCHOLIUM. The statement that mutually equiangular spherical △ are mutually equilateral, and equal, or symmetrical and equivalent, is true only when limited to the same sphere, or equal spheres. But when the spheres are unequal, the spherical △ are similar, but not equal. Hence, to compare two similar spherical △, it is necessary to know the linear extent of two homologous sides; or, what is equivalent, to know the radii of the spheres. And, as in the case of plane △, two similar spherical △ have the same ratio as the squares of the linear measures of any two homologous sides, and therefore as the squares of the radii of the spheres.

ON COMPARISON AND MEASUREMENT OF SPHERICAL SURFACES.

751. DEF. A *Lune* is a part of the surface of a sphere included between two semi-circumferences of great circles.

752. DEF. The *Angle* of a lune is the angle included by the semi-circumferences which forms its boundary. Thus ∠ CAB is the angle of the lune.



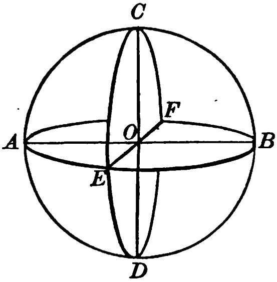
753. DEF. A *Spherical Ungula*, or *Wedge*, is a part of a sphere bounded by a lune and two great semicircles.

754. DEF. The *Base* of an ungula is the bounding lune.

755. DEF. The *Angle* of an ungula is the dihedral of its bounding semicircles, and is equal to the angle of the bounding lune.

756. DEF. The *Edge* of an ungula is the edge of its angle.

757. DEF. The *Spherical Excess* of a spherical triangle is the excess of the sum of its angles over two right angles.



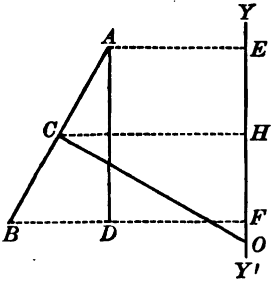
758. DEF. Three planes which pass through the centre of the sphere, each perpendicular to the other two, divide the surface of the sphere into *eight tri-rectangular triangles*. Thus the three planes $CADB$, $CEDF$ and $AEBF$ divide the surface of the sphere into the eight tri-rectangular triangles CEB, DEB, CBF, DBF, etc.

As in Plane Geometry the whole angular magnitude about any point in a plane is divided by two straight lines perpendicular to each other into four right angles, and each right angle is measured by a quadrant, or fourth part of a circumference described about that point as a centre with any given radius; so, if, through a point in space, three planes be made to pass perpendicular to one another, they will divide the whole angular magnitude about that point into *eight solid right angles*, each of which is measured by an eighth part of the surface of a sphere described about that point with any given radius.

And, as in Plane Geometry, each quadrant which measures a right angle is divided into 90 equal parts called degrees, so each of the eight tri-rectangular spherical triangles is divided into 90 equal parts called *degrees of surface*. Hence, the whole surface of the sphere is divided into 720 degrees of surface.

PROPOSITION XXIX. LEMMA.

759. *The area of the surface generated by the revolution of a straight line about another line in the same plane with it as an axis, is equal to the product of the projection of the line on the axis by the circumference whose radius is perpendicular to the revolving line erected at its middle point and terminated by the axis.*



Let the straight line AB revolve about the axis YY' in the same plane; let EF be its projection on the axis; and CO the perpendicular to AB at its middle point C, and terminated in the axis.

We are to prove area $AB = EF \times 2\pi OC$.

The surface generated by AB is the lateral surface of the frustum of a cone of revolution.

Draw $CH \perp$, and $AD \parallel$, to YY'.

Then area $AB = AB \times 2\pi CH$, § 662

(*the lateral area of a frustum of a cone of revolution is equal to the slant height multiplied by the circumference of a section equidistant from its bases*).

The $\triangle$s ABD and COH are similar; § 287

$\therefore AD : AB :: CH : CO$.

But $CH : CO :: 2\pi CH : 2\pi CO$, § 375

(*circumferences of ⊙s have the same ratio as their radii*).

$\therefore AD : AB :: 2\pi CH : 2\pi CO$.

$\therefore AD \times 2\pi CO = AB \times 2\pi CH$.

$\therefore$ area of $AB = AD \times 2\pi CO$.

Now $AD = EF$. § 135

$\therefore$ area $AB = EF \times 2\pi CO$.

Q. E. D.

760. SCHOLIUM. If either extremity of AB be in the axis YY', AB generates the lateral surface of a cone of revolution; and if AB be parallel to the axis YY', it generates the lateral area of a cylinder of revolution. In either case the formula holds good.

EXERCISES.

1. If, from the extremities of one side of a spherical triangle, two arcs of great circles be drawn to a point within the triangle, the sum of these arcs is less than the sum of the other two sides of the triangle.

2. On the same sphere, or on equal spheres, if two spherical triangles have two sides of the one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first will be greater than the third side of the second.

3. To draw an arc perpendicular to a given spherical arc, from a given point without it.

4. At a given point in a given arc, to construct a spherical angle equal to a given spherical angle.

5. To inscribe a circle in a given spherical triangle.

6. Given a spherical triangle whose sides are 60°, 80°, and 100°; find the angles of its polar triangle.

7. The volume of a pyramid is 200 cubic feet; find the volume of a similar pyramid which is three times as high.

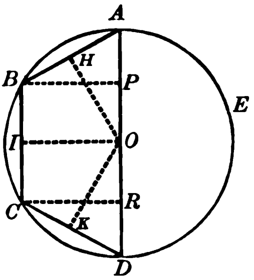
8. Find the centre of a sphere whose surface shall pass through three given points, and shall touch a given plane.

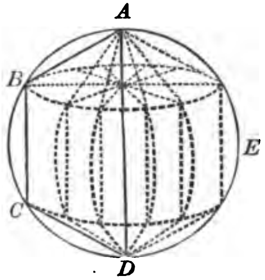
9. Find the centre of a sphere whose surface shall pass through three given points, and shall also touch the surface of a given sphere.

10. Find the centre of a sphere whose surface shall touch two given planes, and also pass through two given points which lie between the planes.

PROPOSITION XXX. THEOREM.

761. *The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*





Let $ABCDE$ be the circumference of a great circle, and AD the diameter, and OA the radius of a sphere.

We are to prove *surface of sphere* $= AD \times 2\pi OA$.

Let the semicircle and any regular inscribed semi-polygon revolve together about the diameter AD.

The semi-circumference will generate the surface of the sphere,

and the semi-perimeter a surface equal to the sum of the surfaces generated by the sides AB, BC, CD, etc.

Draw from the centre O, ⊥s OH, OI and OK to the chords AB, BC, CD, etc.

These ⊥s bisect the chords and are equal; § 185

$\therefore$ area $AB = AP \times 2\pi OH$; § 759

area $BC = PR \times 2\pi OI$;

and area $CD = RD \times 2\pi OK$.

Adding, and observing that $O H$, $O I$ and $O K$ are equal,

$$\text{area } A B C D = (A P + P R + R D) \times 2 \pi O H.$$

$$\therefore \text{area } A B C D = A D \times 2 \pi O H.$$

Now, if the number of sides of the regular inscribed semi-polygon be indefinitely increased, the surface generated by the semi-perimeter will approach the surface of the sphere as its limit, and $O H$ will approach $O A$ as its limit.

$\therefore$ at the limit we have

$$\text{surface of the sphere} = A D \times 2 \pi O A. \qquad \S\, 199$$

Q. E. D.

762. COROLLARY 1. If R denote the radius of the sphere, then $A D$ will equal $2 R$, and $O A$ will equal R. Hence *the surface of a sphere equals* $2 R \times 2 \pi R = 4 \pi R^2$.

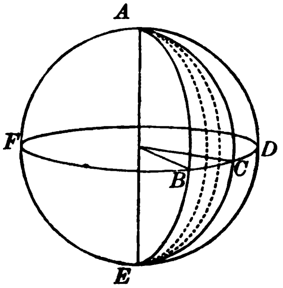
763. COR. 2. Since the area of a great circle of a sphere is equal to πR^2 (§ 381), and the area of the surface of a sphere is equal to $4 \pi R^2$, *the surface of a sphere is equal to four great circles.*

764. COR. 3. If we denote the surfaces of two spheres by S and S', and their radii by R and R', we have $S : S' :: 4 \pi R^2 : 4 \pi R'^2$, or $S : S' :: R^2 : R'^2$; that is, *the surfaces of two spheres have the same ratio as the squares on their radii.*

765. COR. 4. Since $S = 4 \pi R^2 = \pi (2 R)^2$, *the surface of a sphere is equivalent to a circle whose radius is equal to the diameter of the sphere.*

PROPOSITION XXXI. THEOREM.

766. *A lune is to the surface of the sphere as the angle of the lune is to four right angles.*



Let L denote the lune $ABEC$ whose angle is A; S, the surface of the sphere; and $BCDF$, a great circle whose pole is A.

We are to prove $\frac{L}{S}=\frac{A}{4 \text{ rt. } \measuredangle s}$.

Now the arc BC measures the $\angle A$ of the lune; § 714
and the circumference $BCDF$ measures 4 rt. $\measuredangle s$.

CASE I. — *If BC and $BCDF$ be commensurable.*

Find a common measure of BC and $BCDF$.

Suppose this common measure to be contained in BC 3 times, and in $BCDF$ 25 times.

Then $$\frac{A}{4 \text{ rt. } \measuredangle s}=\left(\frac{BC}{BCDF}\right)=\frac{3}{25}.$$

Pass arcs of great ⓢ through A and these points of division.

The entire surface will be divided into 25 equal lunes, of which lune L will contain 3.

$$\therefore \frac{L}{S}=\frac{3}{25}.$$

But $\frac{A}{4 \text{ rt. } \measuredangle s}=\frac{3}{25}$, $\therefore \frac{L}{S}=\frac{A}{4 \text{ rt. } \measuredangle s}$.

CASE II. — *If BC and $BCDF$ be incommensurable,*

the proposition can be proved by the method of limits, as employed in § 201.

Q. E. D.

767. COROLLARY. If we denote the surface of the tri-rectangular triangle by T, the surface of the whole sphere will be $8\ T$ (§ 758), and if we denote the surface of the lune by L, and its angle by A, the unit of the angle being a right angle, we shall have $\frac{L}{8\ T} = \frac{A}{4}$. Therefore $L = T \times 2\ A$.

And if we take the tri-rectangular triangle as the unit of surface in comparing surfaces on the same sphere, we shall have $L = 2\ A$. That is, *if a right angle be the unit of angles and the tri-rectangular triangle be the unit of spherical surfaces, the area of a lune is expressed by twice its angle.*

768. SCHOLIUM. *We may also obtain the area of a lune whose angle is known, on a given sphere, by finding the area of the sphere, and multiplying this area by the ratio of the angle of the lune, expressed in degrees, to* 360°. Thus, if the angle of the lune be 60°, the area of the lune will be $\frac{60}{360}$ of the area of the sphere.

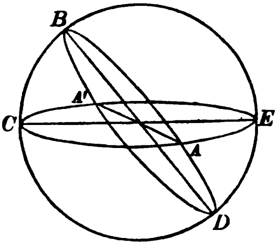
Ex. 1. Given the radius of a sphere is 10 feet; find the area of a lune whose angle is 30°.

2. Given the diameter of a sphere is 16 feet; find the area of a lune whose angle is 75°.

3. Given the diameter of a sphere is 20 inches; find the entire surface of its circumscribed cylinder; and of its circumscribed cone, the vertical angle of the cone being 60°.

PROPOSITION XXXII. THEOREM.

769. *If two circumferences of great circles intersect on the surface of a hemisphere, the sum of the opposite triangles thus formed is equivalent to a lune whose angle is equal to that included by the semi-circumferences.*



Let the semi-circumferences BAD and CAE intersect at A on the surface of a hemisphere.

We are to prove $\triangle ABC + \triangle DAE$ *equivalent to a lune whose angle is* BAC.

The semi-circumferences produced intersect on the opposite hemisphere at A'.

Then each of the arcs AD and $A'B$ is the supplement of AB,

(two great ⓢ bisect each other).

$$\therefore AD = A'B.$$

In like manner, $AE = A'C$ and $DE = BC$.

$\therefore$ $\triangle$s ADE and $A'BC$ are symmetrical and equivalent. § 743

$\therefore \triangle ABC + \triangle ADE = \triangle ABC + \triangle A'BC =$ lune $ABA'CA$.

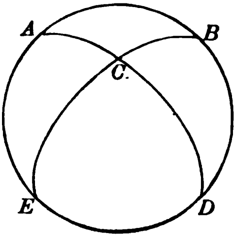
That is, $\triangle ABC + \triangle ADE =$ lune whose $\angle$ is BAC.

Q. E. D.

770. COROLLARY. The sum of two spherical pyramids, the sum of whose bases is equivalent to a lune, is equivalent to a wedge whose base is the lune.

PROPOSITION XXXIII. THEOREM.

771. *The area of a spherical triangle is equal to its spherical excess multiplied by the area of the tri-rectangular triangle.*



Let ABC be a spherical triangle, and T the area of the tri-rectangular triangle.

We are to prove $\triangle ABC = T(\measuredangle A + B + C - 2)$.

Complete the circumference $ABDE$.

Produce AC and BC to meet this circumference in D and E.

Then $\triangle\!\!\!\triangle\, ABC + BCD$ (= lune A) $= T \times 2 \angle A$. § 767

$\triangle\!\!\!\triangle\, ABC + ACE$ (= lune B) $= T \times 2 \angle B$, § 767

$\triangle\!\!\!\triangle\, ABC + DCE$ (= lune C) (§ 769) $= T \times 2 \angle C$. § 767

By adding these equalities,

$$2\triangle ABC + \triangle\!\!\!\triangle\, ABC + BCD + ACE + DCE$$
$$= T \times 2(\measuredangle A + B + C).$$

But $\triangle\!\!\!\triangle\, ABC + BCD + ACE + DCE = 4T$, § 758

(the surface of a hemisphere is equal to 4 tri-rectangular $\triangle\!\!\!\triangle$*).*

$$\therefore 2\triangle ABC + 4T = T \times 2(\measuredangle A + B + C);$$
$$\therefore \triangle ABC = T \times (\measuredangle A + B + C - 2).$$

Q. E. D.

772. SCHOLIUM 1. If $\angle A = 140°$, $\angle B = 120°$ and $\angle C = 100°$, a right angle being the unit,

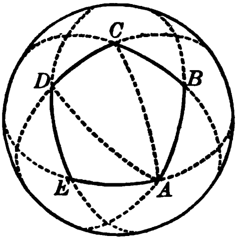
then, $\triangle ABC = T\left(\frac{140°}{90°} + \frac{120°}{90°} + \frac{100°}{90°} - 2\right) = 2T$.

773. SCHO. 2. *To find the area of a spherical triangle on a given sphere, the angles of the triangle being given, we may multiply the area of the hemisphere by the ratio of the spherical excess to* 360°.

Thus if $\angle A = 140°$, $\angle B = 120°$ and $\angle C = 100°$, since the hemisphere is $2\pi R^2$, we have $\triangle ABC = 2\pi R^2 \times \frac{\angle A + \angle B + \angle C - 180°}{360°} = 2\pi R^2 \times \frac{180°}{360°} = \pi R^2$.

PROPOSITION XXXIV. THEOREM.

774. *The area of a spherical polygon is equal to its spherical excess multiplied by the area of the tri-rectangular triangle.*



Let P denote the area of the spherical polygon; S the sum of its angles; n the number of its sides; t, t', t'' ... the areas of the triangles formed by drawing diagonals from any vertex A; s, s', s'' ... respectively the sums of the angles of these triangles; and T the tri-rectangular triangle.

We are to prove $P = T[S - 2(n-2)]$.

Now $t = T(s - 2)$, § 771

(*the area of a spherical* △ *is equal to its spherical excess multiplied into the area of the tri-rectangular* △).

$t' = T(s' - 2)$, § 771

and $t'' = T(s'' - 2), \ldots$

By adding these equalities,

$t + t' + t'', \ldots = T[s + s' + s'' + \ldots - 2(n-2)]$.

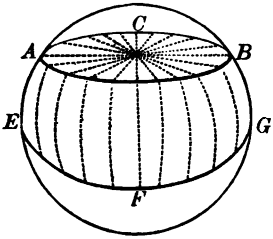
But $t + t' + t'' + \ldots = P$;

and $s + s' + s'' + \ldots = S$.

$\therefore P = T[S - 2(n-2)]$.

Q. E. D.

775. COROLLARY. *The volume of a spherical pyramid is to the volume of the tri-rectangular pyramid, as the base of the pyramid is to the tri-rectangular triangle.* And, since the volume of the tri-rectangular pyramid is $\frac{1}{8}$ the volume of the sphere, and the area of the tri-rectangular triangle is $\frac{1}{8}$ of the surface of the sphere; *the volume of a spherical pyramid is to the volume of the sphere as its base is to the surface of the sphere.*



776. DEF. A *Zone* is the part of the surface of a sphere included between two parallel circles of the sphere; as the surface included between the circles ABC and EFG.

777. DEF. The *Bases* of a zone are the circumferences of the intercepting circles; as circumferences ABC and EFG. If the plane of one base become tangent to the sphere, that base becomes a point, and the zone will have but one base.

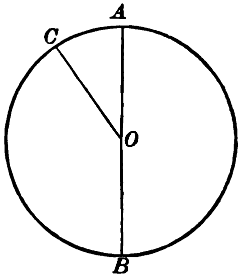
778. DEF. The *altitude* of a zone is the perpendicular distance between the planes of its bases.

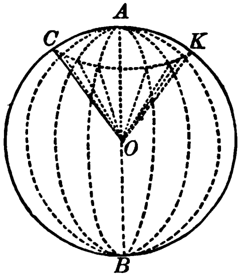
779. DEF. A *Spherical Segment* is a part of the sphere included between two parallel planes.

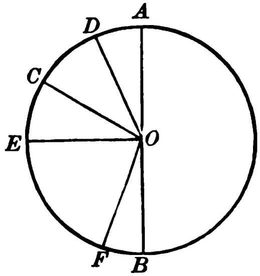
780. DEF. The *Bases* of a spherical segment are the bounding circles.

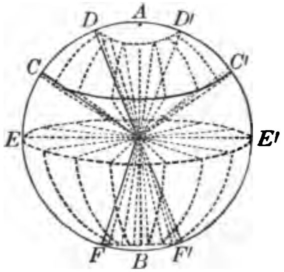
One of the planes may become a tangent plane to the sphere. In this case the segment has but one base.

781. DEF. The *Altitude* of a spherical segment is the perpendicular distance between the planes of its bases.









782. Def. A *Spherical Sector* is a part of a sphere generated by a circular sector of the semicircle which generates the sphere; as $A\ O\ C\ K$.

783. Def. The *Base* of a spherical sector is the zone generated by the arc of the circular sector; as $A\ C\ K$.

The other bounding surfaces of a spherical sector may be one conical surface, or two conical surfaces; or one conical and one plane surface.

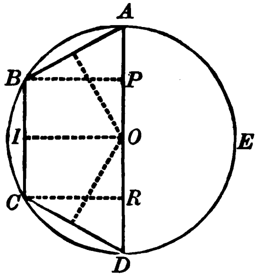
Thus, let $A\ B$ be the diameter around which the semicircle $A\ C\ B$ revolves to generate the sphere. The solid generated by the circular sector $A\ O\ C$ will be a spherical sector having the zone $A\ C\ K$ for its base, and for its other bounding surface the conical surface generated by $C\ O$.

The spherical sector generated by $C\ O\ D$ has for its base the zone generated by $C\ D$, and for its other surfaces the concave conical surface generated by $D\ O$, and the convex conical surface generated by $C\ O$.

The spherical sector generated by $E\ O\ F$ has for its base the zone generated by $E\ F$, and for one surface the plane surface generated by $E\ O$, and for the other surface the concave conical surface generated by $F\ O$.

PROPOSITION XXXV. THEOREM.

784. *The area of a zone is equal to the product of its altitude by the circumference of a great circle.*



Let $ABCDE$ be the circumference of a great circle, BC any arc of this circumference, and OA the radius of the sphere. And, let PR be the altitude of the zone generated by arc BC.

We are to prove zone $BC = PR \times 2\pi OA$.

If the semicircle $ABCD$ revolve about the diameter AD as an axis, the semi-circumference $ABCD$ will generate the surface of a sphere; the arc BC, a zone,

and the chord BC, a surface whose area is $PR \times 2\pi OI$. §759

Now if we bisect the arc BC, and continue this process indefinitely, the surface generated by the chords of these arcs will approach the zone as its limit;

the $\perp$ OI will approach the radius of the sphere as its limit;

while PR will remain constant.

$\therefore$ at the limit, zone $BC = PR \times 2\pi OA$.

Q. E. D.

785. COROLLARY 1. *Zones on the same sphere, or equal spheres, have the same ratio as their altitudes.*

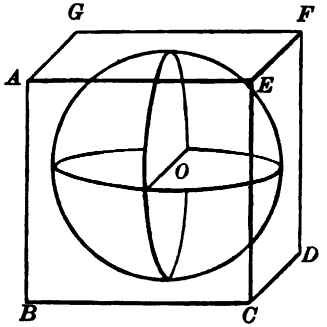
786. COR. 2. *A zone is to the surface of the sphere as the altitude of the zone is to the diameter of the sphere.*

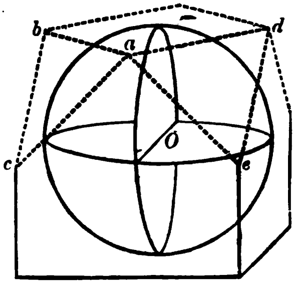
787. COR. 3. Let arc AB generate a zone of a single base. Then, zone $AB = AP \times 2\pi OA$. Hence, zone $AB = \pi AP \times AD = \pi \overline{AB}^2$. (§ 307.) That is, *a zone of one base is equivalent to a circle whose radius is the chord of the generating arc.*

On the Volume of the Sphere.

Proposition XXXVI. Theorem.

788. *The volume of a sphere is equal to the area of its surface multiplied by one-third of its radius.*





Let R be the radius of a sphere whose centre is O, S its surface, and V its volume.

We are to prove $V = S \times \frac{1}{3} R$.

Conceive a cube to be circumscribed about the sphere.

From O, the centre of the sphere, conceive lines to be drawn to the vertices of each of the polyhedral $\angle$s A, B, C, D, etc.

These lines are the edges of six quadrangular pyramids, whose bases are the faces of the cube, and whose common altitude is the radius of the sphere.

The volume of each pyramid is equal to the product of its base by $\frac{1}{3}$ its altitude. § 574

$\therefore$ the volume of the six pyramids, that is, the volume of the circumscribed cube, is equal to the surface of the cube multiplied by $\frac{1}{3} R$.

Now conceive planes drawn tangent to the sphere, cutting each of the polyhedral $\angle$s of the cube.

We shall then have a circumscribed solid whose volume will be nearer that of the sphere than is the volume of the circumscribed cube.

From O conceive lines to be drawn to each of the polyhedral ∡s of the solid thus formed, a, b, c, etc.

These lines will form the edges of a series of pyramids, whose bases are the surface of the solid, and whose common altitude is the radius of the sphere;

and the volume of each pyramid thus formed is equal to the product of its base by $\frac{1}{3}$ its altitude.

∴ the sum of the volumes of these pyramids, that is, the volume of this new solid, is equal to the surface of the solid multiplied by $\frac{1}{3} R$.

Now, this process of cutting the polyhedral ∡s by tangent planes may be considered as continued indefinitely,

and, however far this process is carried, it will always be true that the volume of the solid is equal to its surface multiplied by $\frac{1}{3} R$.

But the sphere is the limit of this circumscribed solid.

$$\therefore V = S \times \tfrac{1}{3} R. \qquad \S\ 199$$

Q. E. D.

789. Corollary 1. Since $S = 4\pi R^2$ (§ 762), $V = 4\pi R^2 \times \frac{1}{3} R = \frac{4}{3}\pi R^3$. If we denote the diameter of the sphere by D, $R^3 = \left(\frac{D}{2}\right)^3 = \frac{D^3}{8}$. $\therefore V = \frac{1}{6}\pi D^3$.

790. Cor. 2. Denote the radius of another sphere by R' and its volume by V'; we have $V' = \frac{4}{3}\pi R'^3$. $\therefore \frac{V}{V'} = \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi R'^3} = \frac{R^3}{R'^3}$.

That is, *spheres are to each other as the cubes of their radii.*

791. Cor. 3. *The volume of a spherical sector is equal to the product of the area of the zone which forms its base by one-third the radius of the sphere.*

Let R denote the radius of a sphere, C the circumference of a great circle, H the altitude of the zone, Z the surface of the zone, and V the volume of the corresponding sector.

Then $$C = 2\pi R; \qquad \text{§ 381}$$

$$Z = C \times H = 2\pi R \times H; \qquad \text{§ 784}$$

$$V = \tfrac{1}{3} Z \times R = \tfrac{2}{3}\pi R^2 \times H.$$

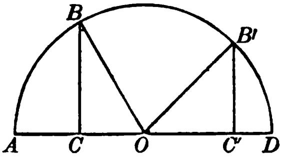
792. Cor. 4. *The volumes of spherical sectors of the same sphere, or equal spheres, are to each other as the zones which form their bases, or as the altitudes of these zones.*

For, let V and V' denote the volumes of two spherical sectors, Z and Z' the zones which form their bases, H and H' the altitudes of these zones, and R the radius of the sphere.

Then $$\frac{V}{V'} = \frac{Z \times \frac{1}{3}R}{Z' \times \frac{1}{3}R} = \frac{Z}{Z'}.$$

And since $$\frac{Z}{Z'} = \frac{H}{H'}, \qquad \text{§ 785}$$

$$\frac{V}{V'} = \frac{H}{H'}.$$



793. Cor. 5. The volume of a spherical segment of one base, less than a hemisphere, generated by the revolution of a circular sector $A\ O\ B$ about the diameter $A\ D$, may be found by subtracting the volume of the cone of revolution generated by $O\ B\ C$ from that of the spherical sector $A\ O\ B$.

In like manner, the volume of a spherical segment of one base, greater than a hemisphere, generated by the revolution of

$A\,B'C'$ may be found by adding the volume of the cone of revolution generated by $O\,B'\,C'$ to that of the spherical sector generated by $A\,O\,B'$.

794. Cor. 6. The volume of a spherical segment of two bases, generated by the revolution of $C\,B\,B'\,C'$ about the diameter $A\,D$, may be found by subtracting the volume of the segment of one base generated by $A\,B\,C$ from that of the segment of one base generated by $A\,B'\,C'$.

Exercises.

1. Given a sphere whose diameter is 20 inches; find the circumference of a small circle whose plane cuts the diameter 4 inches from the centre.

2. Construct, on the spherical blackboard, spherical angles of 30°, 45°, 90°, 120°, 150° and 135°.

3. Construct, on the spherical blackboard, a spherical triangle, whose sides are 100°, 80° and 70° respectively. What is true of its polar triangle?

4. Find the surface and volume of a sphere whose radius is 10 inches; also find the area of a spherical triangle on this sphere, the angles of the triangle being 80°, 85° and 100° respectively.

5. If 7 equidistant planes cut a sphere, each perpendicular to the same diameter, what are the relative areas of the zones?

6. Given, two mutually equiangular triangles on spheres whose radii are 10 inches and 40 inches respectively; what are their relative areas?

7. Let V denote the volume of a spherical pyramid, S its base, E the spherical excess of its base, and R the radius of the sphere; show that $S = \frac{1}{2}\,\pi\,R^2\,E$, and $V = \frac{1}{6}\,\pi\,R^3\,E$.

8. Given, the volume of a sphere 1728 inches: find its radius.

9. Find the ratio of the surfaces, and the ratio of the volumes, of a cube and of the inscribed sphere.

10. Find the ratio of the surfaces, and the ratio of the volumes, of a sphere and the circumscribed cylinder.

11. Let V denote the volume and H the altitude of the spherical segment of one base, and R the radius of the sphere; show that $V = \pi H^2 (R - \frac{1}{3} H)$. Also, find V when $R = 12$ and $H = 3$.

12. Given, a sphere 2 feet in diameter; find the volume of a segment of the sphere included between two parallel planes, one at 3 and the other at 9 inches from the centre. (Two solutions.)

13. A sphere 4 inches in diameter is bored through the centre with a two-inch auger; find the volume remaining.

THE END.